HYPER-ALGEBRAIC INVARIANTS OF p-ADIC ALGEBRAIC NUMBERS

SHANWEN WANG AND YIJUN YUAN

ABSTRACT. Let $p \geq 3$ be a prime. In this article, we introduce two arithmetic invariants (hyper-tame indexes and hyper-inertia indexes) of the hyper-algebraic elements in the p-adic Mal'cev-Neumann field $\mathbb{L}_p$. For p-adic algebraic numbers that generate abelian extensions and tamely ramified extensions of $\mathbb{Q}_p$, we calculate their hyper-tame indexes and hyper-inertia indexes.

CONTENTS

1. INTRODUCTION

Let $p \geq 3$ be a prime throughout this article. In [Poo93], the p-adic Mal'cev-Neumann field $\mathbb{L}_p := W(\overline{\mathbb{F}}_p)((p^{\mathbb{Q}}))$ is constructed and a necessary condition for an element in $\mathbb{L}_p$ to be algebraic over $\mathbb{Q}_p$ is given. More precisely, an element $f \in \mathbb{L}_p$ can be written uniquely in the form

$$f = \sum_{q \in \mathbb{Q}} [r_q] p^q,$$

with $r_q \in \overline{\mathbb{F}}_p$ and $\mathrm{supp}(f) = \{q \in \mathbb{Q} \colon r_q \neq 0\}$ a well-ordered subset of $\mathbb{Q}$; thus, an element $f = \sum_{q \in \mathbb{Q}} [r_q] p^q \in \mathbb{L}_p$ is completely determined by its support and its coefficients. As stated in [Poo93, Corollary 8], if f is algebraic, then it satisfies the following conditions:

(1) there exists a positive integer N that $\mathrm{supp}(f) \subseteq \frac{1}{N}\mathbb{Z}[1/p]$;
(2) there exists a positive integer k such that $r_q \in \mathbb{F}_{p^k}$ for all $q \in \mathrm{supp}(f)$.

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An element $f \in \mathbb{L}_p$ satisfying the above conditions is called *hyper-algebraic*. The set $\mathbb{L}_p^{\mathrm{ha}}$ of hyper-algebraic elements in $\mathbb{L}_p$ forms an algebraically closed field containing $\mathbb{Q}_p$. As a result, all p-adic algebraic numbers are hyper-algebraic, i.e. $\overline{\mathbb{Q}}_p \subseteq \mathbb{L}_p^{\mathrm{ha}}$.

The first result of this article is a clarification of relations among the fields $\mathbb{L}_p^{\mathrm{ha}}$, $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$:

Theorem A (cf. Theorem 3.3)**.** *The field $\mathbb{L}_p^{\mathrm{ha}}$ is strictly larger than $\overline{\mathbb{Q}}_p$ and it is neither complete nor a subfield of $\mathbb{C}_p$.*

This leads us to study the behavior of p-adic algebraic numbers in $\mathbb{L}_p^{\mathrm{ha}}$. We introduce two invariants of a hyper-algebraic element θ: hyper-tame index $\mathfrak{T}_\theta$ (i.e.the minimal positive integer e such that $\mathrm{supp}(\theta) \subseteq \frac{1}{e}\mathbb{Z}[1/p]$) and hyper-inertia index $\mathfrak{F}_\theta$ (i.e. the minimal positive integer f such that $r_q \in \mathbb{F}_{p^f}$ for all $q \in \mathrm{supp}(\theta)$), and we use them to describe abelian extensions and tamely ramified extensions of $\mathbb{Q}_p$.

Theorem B (cf. Theorem 4.5)**.** *Let $\alpha \in \overline{\mathbb{Q}}_p$ be a p-adic algebraic number with $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ an abelian extension of degree n. Denote by $\mathbf{f}_{\mathbb{Q}_p(\alpha)}$ the local conductor of $\mathbb{Q}_p(\alpha)$ over $\mathbb{Q}_p$. Then*

(1) If $\mathbf{f}_{\mathbb{Q}_p(\alpha)} = 0$, then $\mathfrak{T}_\alpha = 1$ and $\mathfrak{F}_\alpha = n$.
(2) If $\mathbf{f}_{\mathbb{Q}_p(\alpha)} \geq 1$, then $\mathfrak{T}_\alpha \mid p-1$ and

$$\mathfrak{F}_\alpha \mid \begin{cases} \mathrm{lcm}(2, n), & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} = 1, 2; \\ \mathrm{lcm}\Big(2 \cdot p^{\mathbf{f}_{\mathbb{Q}_p(\alpha)}-1}, n\Big), & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} \geq 3. \end{cases}.$$

Remark 1.1. *For $\alpha \in \mathbb{L}_p$, we denote by $[C_{\frac{1}{p-1}}(\alpha)]$ the coefficient of index $\frac{1}{p-1}$ of the canonical expansion of α. Based on our computation of the truncated expansion of ζ_{p^n} (cf. Example 2.13), we state a conjecture on $C_{\frac{1}{p-1}}(\zeta_{p^n})$: for any integer $n \geq 2$ and p^n-th primitive root of unity ζ_{p^n}, there exists another p^n-th primitive root of unity ζ'_{p^n} with $C_{\frac{1}{p-1}}(\alpha) = 0$ such that ζ_{p^n}/ζ'_{p^n} is a p^{n-1}-th root of unity (not necessarily primitive).*

If this conjecture holds, then $\mathfrak{F}_{\zeta_{p^n}} = 2$ for every $n \geq 2$, and consequently $\mathfrak{F}_\alpha$ divides $\mathrm{lcm}(2, n)$ for all ramified cases in the above theorem. See the proof of Proposition 4.1 for more details. Note that this conjecture is true when $n = 2$ (cf. Lemma 4.3).

Definition 1.2. *Let K be a finite extension of $\mathbb{Q}_p$.*

*(1) Denote by $\mathfrak{f}_K$ the **inertia degree** of K over $\mathbb{Q}_p$.*
*(2) Denote by $\mathfrak{e}_K$ the **ramification index** of K over $\mathbb{Q}_p$ and by $\mathfrak{e}_K^{\mathrm{t}}$ the **tame ramification index** of K over $\mathbb{Q}_p$ respectively, i.e. the prime-to-p part of $\mathfrak{e}_K$.*
(3) For any p-adic algebraic number α, we denote by $\mathfrak{f}_\alpha$ (resp. $\mathfrak{e}_\alpha$, $\mathfrak{e}_\alpha^{\mathrm{t}}$) for $\mathfrak{f}_{\mathbb{Q}_p(\alpha)}$ (resp. $\mathfrak{e}_{\mathbb{Q}_p(\alpha)}$, $\mathfrak{e}^{\mathrm{t}}_{\mathbb{Q}_p(\alpha)}$).

In [Lam86], Lampert remarks that if $\mathbb{Q}_p(\alpha)$ is tamely ramified over $\mathbb{Q}_p$, then $\mathrm{supp}(\alpha)$ is contained in $\frac{1}{\mathfrak{e}_\alpha^{\mathrm{t}}}\mathbb{Z}$. The following theorem refines this result:

Theorem C (cf. Theorem 4.7)**.** *Let $\alpha \in \mathbb{L}_p^{\mathrm{ha}}$ be a hyper-algebraic element in $\mathbb{L}_p$. Then $\mathbb{Q}_p(\alpha)$ is tamely ramified over $\mathbb{Q}_p$ if and only if $\mathrm{supp}(\alpha) \subseteq \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}$. In this situation, we have $\mathfrak{T}_\alpha = \mathfrak{e}_\alpha^{\mathrm{t}}$, $\mathfrak{f}_\alpha \mid \mathfrak{F}_\alpha$ and $\mathfrak{F}_\alpha \mid \mathrm{ord}_{\mathfrak{e}_\alpha^{\mathrm{t}}(p^{\mathfrak{f}_\alpha}-1)}\, p$.*

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2. Prelimiaries on valued fields

2.1. **Maximally complete fields and Mal'cev-Neumann fields.** The main objective of this subsection is to justify the notion of immediate maximally complete of a valued field, in particular, of the field $\mathbb{C}_p$ of p-adic complex numbers.

Definition 2.1. *Let (F, v) be a valued field.*

(1) *Say (E, w) is an* ***immediate extension*** *of F if it is an extansion of (F, v) and has the same value group and residue field as F.*
(2) *Say (F, v) is* ***maximally complete*** *if it has no proper immediate extension.*

Unsurprisingly, one has the following result

Proposition 2.2 ([Poo93, Proposition 6])**.**

(1) *Maximally complete fields are complete.*
(2) *If a maximally complete field has divisible value group and algebraically closed residue field, then itself is algebraically closed.*

Remark 2.3.

(1) *The proof of this Proposition, which is due to MacLane, is not effective, i.e. it does not give an algorithm to construct a root of a given polynomial over F.*
(2) *Kaplansky showed in [Kap42, Section 5] that there exist valued fields with two immediate maximally complete extensions that are not isomorphic as fields.*

Definition 2.4. *Let F be a valued field and (E_1, w_1), (E_2, w_2) be two extension of F.*

(1) *Say E_1 and E_2 are* ***analytically equivalent*** *if there exists a F-isomorphism of field $i\colon E_1 \longrightarrow E_2$ such that $w_2(i(x)) = w_1(x)$ for any $x \in E_1$.*
(2) *Say E_1 embeds into E_2 if E_1 is analytically equivalent to a subfield of E_2.*

Theorem 2.5 ([Poo93, Corollary 6])**.** *Every valued field F has an immediate maximally complete extension. If F has divisible value group and algebraically closed residue field, then the immediate maximally complete extension is unique up to analytic equivalence.*

By Theorem 2.9, a standard way to produce maximally complete fields is to consider the Mal'cev-Neumann fields which we recall in the following.

Definition 2.6 ([Poo93, Section 3])**.** *Let R be a commutative ring and G be an ordered group.*

(1) *For any $f \in \mathrm{Hom}_{\mathrm{Set}}(G, R)$, we define the* ***support*** *of f to be*
$$\mathrm{supp}(f) = \{g \in G\colon f(g) \neq 0\}.$$
(2) *Define the set of* ***Mal'cev-Neumann series*** *over R with value group G to be*
$$R(\!(G)\!) := \{f \in \mathrm{Hom}_{\mathrm{Set}}(G, R)\colon \mathrm{supp}(f) \textit{ is well-ordered}\}.$$
By introducing a formal variable t, elements in $R(\!(G)\!)$ will also be written as $\sum_{g\in G} r_g t^g$, where $r_g \in R$ for all $g \in G$.

Proposition 2.7 ([Poo93, Lemma 1, Corollary 2]). *Let R be a commutative ring and G be an ordered group.*

(1) With identity $1 \cdot t^0$ and addition as well as multiplication given by

$$\sum_{g\in G} b_g t^g + \sum_{g\in G} b_g t^g := \sum_{g\in G}(a_g + b_g)t^g, \ \sum_{g\in G} b_g t^g \cdot \sum_{g\in G} b_g t^g := \sum_{g\in G}\left(\sum_{h\in G} a_h b_{g-h}\right)t^,$$

$R(\!(G)\!)$ forms a commutative ring.

(2) If R is a field, then so does $R(\!(G)\!)$. Moreover, with the map

$$v\colon R(\!(G)\!) \longrightarrow G \cup \{\infty\}, \ f \longmapsto \begin{cases} \min \operatorname{supp}(f), & \text{if } f \neq 0 \\ \infty, & \text{if } f = 0 \end{cases},$$

$R(\!(G)\!)$ becomes a valued field with value group G and residue field R.

Note that $\operatorname{char} R(\!(G)\!) = \operatorname{char} \mathrm{R}$, we call $R(\!(G)\!)$ the **equal-characteristic Mal'cev-Neumann field** over R with value group G, also denoted as $R(\!(t^G)\!)$ with respect to the formal variable t.

Theorem 2.8 ([Poo93, Proposition 3, Corollary 3, Proposition 5]). *Let k be a perfect field of characteristic p and G be an ordered group containing $\mathbb{Z}$ as a subgroup. Besides that, let*

$$\mathcal{N} := \left\{ \sum_{g\in G} r_g t^g \in W(k)(\!(t^G)\!) \colon \text{ for every } g \in G, \ \sum_{n\in\mathbb{Z}} r_{g+n} p^n = 0 \right\},$$

where $W(k)$ is the ring of Witt vectors of k. Then

*(1) $\mathcal{N}$ is a maximal ideal of $W(k)(\!(t^G)\!)$, which makes $W(k)(\!(p^G)\!) := W(k)(\!(t^G)\!)/\mathcal{N}$ a field[1], called the **p-adic Mal'cev-Neumann field**.*

(2) Every element in $W(k)(\!(p^G)\!)$ can be uniquely (and formally) written as

$$\sum_{g\in G} [r_g] p^g,$$

where $r_g \in k$ for all $g \in G$ and $[\cdot]\colon k \longrightarrow W(k)$ is the Teichmüller lift.

*(3) For $f = \sum_{g\in G} [r_g] p^g$, define the **support** of f to be*

$$\operatorname{supp}(f) = \{g \in G \colon r_g \neq 0\}.$$

Then the map

$$v\colon W(k)(\!(G)\!)/\mathcal{N} \longrightarrow G \cup \{\infty\}, \ f \mapsto \begin{cases} \min \operatorname{supp}(f), & \text{if } f \neq 0 \\ \infty, & \text{if } f = 0 \end{cases}$$

makes $W(k)(\!(G)\!)/\mathcal{N}$ a mixed-characteristic valued field with value group G and residue field k.

Theorem 2.9 ([Poo93, Theorem 1]). *The equal-characteristic and p-adic Mal'cev-Neumann fields are maximally complete.*

Theorem 2.10 ([Poo93, Corollary 5, Corollary 6]). *Let F be a valued field with value group G and residue field k with $\operatorname{char} k = 0$ or p. Let $\widetilde{G}$ be a divisible group that contains G.*

[1] Intuitively speaking, $W(k)(\!(p^G)\!)$ is obtained by replacing the formal variable t of elements in $W(k)(\!(t^G)\!)$ by the prime p.

(1) The field F embeds into the Mal'cev-Neumann field

$$\begin{cases} k^{\mathrm{alg}}(\!(t^{\widetilde{G}})\!), & \text{if } \operatorname{char} F = \operatorname{char} k; \\ W(k^{\mathrm{alg}})(\!(p^{\widetilde{G}})\!), & \text{if } \operatorname{char} F \neq \operatorname{char} k; \end{cases}$$

where k^{alg} is an algebraic closure of k.

(2) If $G = \widetilde{G}$ and $k = k^{\mathrm{alg}}$, then the Mal'cev-Neumann field

$$\begin{cases} k(\!(t^{G})\!), & \text{if } \operatorname{char} F = \operatorname{char} k; \\ W(k)(\!(p^{G})\!), & \text{if } \operatorname{char} F \neq \operatorname{char} k; \end{cases}$$

is the unique (up to analytic equivalence) immediate maximally complete extension of F (cf. Theorem 2.5).

Example 2.11. *It is well-known that $\mathbb{C}_p$ is not maximally complete (cf. [BS18, Theorem 4.8, Theorem 6.7]). Since it has value group $\mathbb{Q}$ and residue field $\overline{\mathbb{F}}_p$, we can apply Theorem 2.10 (2) to $\mathbb{C}_p$, which gives its unique immediate maximally complete extension*

$$\mathbb{L}_p := W(\overline{\mathbb{F}}_p)(\!(p^{\mathbb{Q}})\!).$$

By applying Proposition 2.2 to $\mathbb{L}_p$, one knows that $\mathbb{L}_p$ is complete and algebraically closed. Moreover, one can show that $\mathbb{L}_p$ is much larger than $\mathbb{C}_p$:

Lemma 2.12 ([Poo93, Corollary 9])**.** *The field $\mathbb{L}_p$ has transcendence degree $2^{\aleph_0}$ over $\mathbb{C}_p$.*

2.2. **Basic properties of $\mathbb{L}_p$.** Compared to the unsatisfactoriness mentioned in Remark 2.3 (1), Kedlaya proved[2,3] the algebraic closeness of $\mathbb{L}_p$ by using a transfinite Newton algorithm as following:

For a non-constant polynomial $P(T) = \sum_{i=0}^{n} a_{n-i}T^i \in \mathbb{L}_p[T]$, denote by $\mathscr{N}ewt(P)$ the Newton polygon of P, i.e. the lower boundary of the convex hull of the set of points $(i, v_p(a_i))$ for $i = 0, 1, \cdots, n$. We write $s^P_{\max}$ for the slope of the segment of $\mathscr{N}ewt(P)$ with the largest slope and $m^P_{\max}$ the left endpoint of this segment. Besides that, call

$$\operatorname{Res}_P(T) := \sum_{k=0}^{n-m^P_{\max}} C_{v_p(a_m)+s^P_{\max}(n-m^P_{\max}-k)}(a_{n-k})T^k$$

the residue polynomial of P, where for any $s \in \mathbb{Q}$, the map $C_s\colon \mathbb{L}_p \longrightarrow \overline{\mathbb{F}}_p$ is given by $\sum_{q\in\mathbb{Q}}[\zeta_q]p^q \longmapsto \zeta_s$.

We extracted Kedlaya's proof into the following pseudo-code:

Algorithm 1 transfinite Newton algorithm for $\mathbb{L}_p$

INPUT: A non-constant polynomial $P(T) \in \mathbb{L}_p[T]$
OUTPUT: A root of $P(T)$ in $\mathbb{L}_p$

```
r ← 0, Φ(T) ← P(T)          ▷ We denote the coefficient of T^i in Φ as b_{n−i}.
while Φ(0) ≠ 0 do                     ▷ This loop runs transfinitely.
    c ← any root of Res_Φ(T) in F̄_p
    r ← r + [c] · p^{s^Φ_max}
    Φ(T) ← Φ(T + [c] · p^{s^Φ_max})
end while
return r
```

[2]His proof is motivated by the work of Lampert (cf. [Lam86]).

[3]Actually Kedlaya's proof can be adapted to any Mal'cev-Neumann field (equal-characteristic or p-adic) with divisible value group and algebraically closed residue field.

We refer to [WY21] for a full explanation of this algorithm.

Let $r = \sum_{\omega}[\zeta_\omega]p^{k_\omega} \in \mathbb{L}_p$, with ordinal ω runs through the well-ordered set $\mathrm{supp}(r)$, be a root of $P(T)$ given by the above algorithm. For the convenience of later discussion, we call $r_\omega = \sum_{r<\omega}[\zeta_\omega]p^{k_\omega}$ the ω-th approximation of r, $P_\omega = P(T + r_\omega)$ the ω-th approximation polynomial and $\mathrm{Res}_{P_\omega}(T)$ the ω-th residue polynomial.

Example 2.13 ([WY21; WY23])**.** *For integer $n \geq 1$, denote by ζ_{p^n} a p^n-th root of unity in $\mathbb{C}_p$.*

(1) If n=1, then there exist a p-th root of unity, whose expansion in $\mathbb{L}_p$ is given by

$$\zeta_p = \sum_{i=k}^{\infty}[c_k]p^{\frac{k}{p-1}},$$

where $c_k \in \mathbb{F}_{p^2}$.

(2) If $n \geq 2$, then there exists a p^n-th root of unity, whose (non-canonical) expansion in $\mathbb{L}_p$ is partially given by

$$\begin{aligned}\zeta_{p^n} =& \sum_{k=0}^{p-1}\frac{(-1)^{nk}}{k!}\zeta_{2(p-1)}^{k}p^{\frac{k}{p^{n-1}(p-1)}} + \sum_{k=0}^{p-1}\frac{(-1)^{n(k+1)}}{k!}\zeta_{2(p-1)}^{k+1}p^{\frac{k+p}{p^{n-1}(p-1)}}\left(\sum_{l=n}^{\infty}p^{-1/p^l}\right)\\ &- \sum_{k=1}^{p-1}\frac{(-1)^{n(k+1)}}{k!}\left(\sum_{l=1}^{k}\frac{1}{l}\right)\zeta_{2(p-1)}^{k+1}p^{\frac{k+p}{p^{n-1}(p-1)}}\\ &+ \frac{1}{2}\zeta_{2(p-1)}^{2}p^{\frac{2}{p^{n-2}(p-1)}}\left(\sum_{l=n}^{\infty}p^{-1/p^l}\right)^2 + \frac{(-1)^n}{2}\zeta_{2(p-1)}^{3}p^{\frac{2}{p^{n-2}(p-1)}-\frac{p-2}{p^n(p-1)}}\\ &+ \cdots \text{ terms with higher valuation.}\end{aligned}$$

3. Field of hyper-algebraic elements in $\mathbb{L}_p$

3.1. **Hyper-algebraic elements.** A necessary condition for an element in $\mathbb{L}_p$ to be algebraic over $\mathbb{Q}_p$ has already been given by Poonen (cf. [Poo93]), following a remark from Lampert (cf. [Lam86]). Poonen's condition leads to the following definition of hyper-algebraic element in $\mathbb{L}_p$.

Definition 3.1. *We call an element $f = \sum_{q\in\mathbb{Q}}[r_q]p^q \in \mathbb{L}_p$ **hyper-algebraic**, if it satisfies:*

(1) there exists a positive integer N such that $\mathrm{supp}(f) \subseteq \frac{1}{N}\mathbb{Z}[1/p]$;
(2) there exists a positive integer k such that $r_q \in \mathbb{F}_{p^k}$ for all $q \in \mathrm{supp}(f)$.

Denote by $\mathbb{L}_p^{\mathrm{ha}}$ the set of all hyper-algebraic elements in $\mathbb{L}_p$.

Proposition 3.2 (Lampert, Poonen)**.** *The set $\mathbb{L}_p^{\mathrm{ha}}$ forms an algebraically closed field. As a consequence, all p-adic algebraic numbers are hyper-algebraic, i.e. $\overline{\mathbb{Q}}_p \subseteq \mathbb{L}_p^{\mathrm{ha}}$.*

Theorem 3.3. *The field $\mathbb{L}_p^{\mathrm{ha}}$ is strictly larger than $\overline{\mathbb{Q}}_p$, and it is neither complete nor a subfield in $\mathbb{C}_p$.*

Proof. Consider the sequence $\left(\sum_{k=1}^{n}p^{k-1/k}\right)_{n\geq1}$ in $\overline{\mathbb{Q}}_p \subseteq \mathbb{L}_p^{\mathrm{ha}}$, which clearly converges in $\mathbb{C}_p$. However, its limit $\sum_{k=1}^{\infty}p^{k-1/k}$ is not hyper-algebraic in $\mathbb{L}_p$, as the p-power-free part of the denominators of elements of its support is unbounded. This shows that $\mathbb{L}_p^{\mathrm{ha}}$ is not complete and does not contain $\mathbb{C}_p$.

To prove it is not contained in $\mathbb{C}_p$, we can consider the following element of $\mathbb{L}_p^{\mathrm{ha}}$:

$$\alpha = \sum_{k=1}^{\infty}p^{\frac{\lfloor\sqrt{2}\cdot p^k\rfloor}{p^k}}.$$

If $\alpha \in \mathbb{C}_p$, then there exists a p-adic algebraic number $\beta \in \overline{\mathbb{Q}}_p$ that $v_p(\alpha - \beta) > 2$. This shows that the canonical expansion of β in $\mathbb{L}_p^{\mathrm{ha}}$ has the form

$$\beta = \sum_{k=1}^{\infty} p^{\frac{\lfloor \sqrt{2} \cdot p^k \rfloor}{p^k}} + \text{ terms with exponent greater than } 2 \cdots$$

Thus $\operatorname{supp}(\beta)$ has accumulation value $\sqrt{2}$. However this is impossible: Lampert showed in [Lam86, Theorem 2] that the set

$$\mathcal{A} := \{\alpha \in \mathbb{L}_p | \{\text{accumulation value of } \operatorname{supp}(\alpha)\} \subset \mathbb{Q}\}$$

is an algebraically closed field. Since the support of every p-adic rational number lies in $\mathbb{Z} \subset \mathbb{Q}$, $\overline{\mathbb{Q}}_p$ is a subfield of $\mathcal{A}$. On the other hand, β does not belong to $\mathcal{A}$. This is a contradiction. □

3.2. Hyper-tame index and hyper-inertia index.

Definition 3.4. *Let $\theta = \sum_{q \in \mathbb{Q}} [r_q] p^q \in \mathbb{L}_p^{\mathrm{ha}}$ be a hyper-algebraic element in $\mathbb{L}_p$.*

(1) Denote by $\mathfrak{T}_\theta$ the minimal positive integer e such that $\operatorname{supp}(\theta) \subseteq \frac{1}{e}\mathbb{Z}[1/p]$. *We call it the **hyper-tame index** of θ.*
*(2) Denote by $\mathfrak{F}_\theta$ the minimal positive integer f such that $r_q \in \mathbb{F}_{p^f}$ for all $q \in \operatorname{supp}(\theta)$. We call it the **hyper-inertia index** of θ.*

*We call them the **hyper-algebraic invariants** of θ.*

The following lemma collects several basic properties of the hyper-tame and hyper-inertia indices:

Lemma 3.5. *Let $\alpha, \beta \in \mathbb{L}_p^{\mathrm{ha}}$ be two hyper-algebraic elements in $\mathbb{L}_p$. Then one has*

(1) $\mathfrak{T}_{\alpha+\beta} \mid \operatorname{lcm}(\mathfrak{T}_\alpha, \mathfrak{T}_\beta)$, $\mathfrak{F}_{\alpha+\beta} \mid \operatorname{lcm}(\mathfrak{F}_\alpha, \mathfrak{F}_\beta)$.
(2) $\mathfrak{T}_{\alpha\cdot\beta} \mid \operatorname{lcm}(\mathfrak{T}_\alpha, \mathfrak{T}_\beta)$, $\mathfrak{F}_{\alpha\cdot\beta} \mid \operatorname{lcm}(\mathfrak{F}_\alpha, \mathfrak{F}_\beta)$. In particular if α is algebraic over $\mathbb{Q}_p$ and $\mathbb{Q}_p(\alpha)$ is unramified over $\mathbb{Q}_p$, then $\mathfrak{T}_{\alpha\cdot\beta} \mid \mathfrak{T}_\beta$ and $\mathfrak{F}_{\alpha\cdot\beta} \mid \operatorname{lcm}(\mathfrak{f}_\alpha, \mathfrak{F}_\beta)$.
(3) $\mathfrak{T}_{1/\alpha} = \mathfrak{T}_\alpha$, $\mathfrak{F}_{1/\alpha} = \mathfrak{F}_\alpha$ for $\alpha \neq 0$.

Proof. The first and the second assertions follow from the definition of addition and multiplication on $\mathbb{L}_p$. In particular if $\mathbb{Q}_p(\alpha)$ is unramified over $\mathbb{Q}_p$, then $\mathbb{Q}_p(\alpha) = \operatorname{Frac} W(\mathbb{F}_{p^{\mathfrak{f}_\alpha}})$. As a result, every element in $\mathbb{Q}_p(\alpha)$ has the form $\sum_{k \gg -\infty} [\zeta_k] p^k$, where $\zeta_k \in \mathbb{F}_{p^{\mathfrak{f}_\alpha}}$ for all l. This shows that $\mathfrak{T}_\alpha = 1$ and $\mathfrak{F}_\alpha = \mathfrak{f}_\alpha$.

For the third assertion, the result is trivial when $|\operatorname{supp}(\alpha)| = 1$, thus we only focus on the case of $|\operatorname{supp}(\alpha)| \geq 2$. Write $\alpha = [\zeta] p^{v_p(\alpha)} - A$ with $v_p(A) > v_p(\alpha)$. Then $\zeta \in \mathbb{F}_{p^{\mathfrak{F}_\alpha}}$, $\mathfrak{T}_A \mid \mathfrak{T}_\alpha$ and $\mathfrak{F}_A \mid \mathfrak{F}_\alpha$. The result follows from the expansion

$$\alpha^{-1} = [\zeta^{-1}] p^{-v_p(\alpha)} \sum_{k=0}^{\infty} \Big([\zeta^{-1}] p^{-v_p(\alpha)} \cdot A \Big)^k,$$

where

$$v_p\Big([\zeta^{-1}] p^{-v_p(\alpha)} \cdot A \Big) > 0, \ \mathfrak{T}_{[\zeta^{-1}] p^{-v_p(\alpha)} \cdot A} \mid \mathfrak{T}_\alpha \text{ and } \mathfrak{F}_{[\zeta^{-1}] p^{-v_p(\alpha)} \cdot A} \mid \mathfrak{F}_\alpha.$$

□

Corollary 3.6. *For any positive integer $e, f \geq 1$, the set*

$$\mathbb{L}_p^{\mathrm{ha}}(e, f) := \{\alpha \in \mathbb{L}_p^{\mathrm{ha}} \colon \mathfrak{F}_\alpha \mid f, \ \mathfrak{T}_\alpha \mid e\}$$

is a subfield of $\mathbb{L}_p^{\mathrm{ha}}$. In particular, if $\alpha \in \mathbb{L}_p^{\mathrm{ha}}(e, f)$, then $\mathbb{Q}_p(\alpha) \subset \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$.

Proposition 3.7. *For every p-adic algebraic number α, the maximal prime divisor of its hyper-tame index $\mathfrak{T}_\alpha$ (resp. hyper-inertia index $\mathfrak{F}_\alpha$) does not exceed $[\mathbb{Q}_p(\alpha) \colon \mathbb{Q}_p]$.*

Proof. Let $n = [\mathbb{Q}_p(\alpha)\colon \mathbb{Q}_p]$. Let

$$\mathcal{R}_n = \{r \in \mathbb{N}\colon \text{ the prime divisor of } r \leq n\}$$

and

$$\mathcal{E}_n = \mathbb{Z}\left[\frac{1}{k}\colon k \in \mathcal{R}_n\right] = \mathbb{Z}\left[\frac{1}{1}, \frac{1}{2}, \cdots, \frac{1}{n}\right].$$

Then Lemma 3.5 implies that the set

$$\mathbb{L}_p^{\mathrm{ha}}(n) = \{\alpha \in \mathbb{L}_p^{\mathrm{ha}}\colon \mathfrak{T}_\alpha \in \mathcal{E}_n, \mathfrak{F}_\alpha \in \mathcal{R}_n\}$$

is a subfield of $\mathbb{L}_p^{\mathrm{ha}}$. Denote by $\mathrm{Min}_\alpha(T)$ the minimal polynomial of α over $\mathbb{Q}_p \subset \mathbb{L}_p^{\mathrm{ha}}(n)$. Since the denominator of the maximal slope (resp. the degree of the residue polynomial) in each step of the Newton algorithm is bounded by n, one can show by transfinite induction that there exists at least one root β of $\mathrm{Min}_\alpha(T)$ that lies in $\mathbb{L}_p^{\mathrm{ha}}(n)$. By replacing $\mathrm{Min}_\alpha(T)$ with $\mathrm{Min}_\alpha(T)/(T-\beta)$ inductively, one knows that $\alpha \in \mathbb{L}_p^{\mathrm{ha}}(n)$. The result follows. □

4. p-ADIC ALGEBRAIC NUMBERS IN $\mathbb{L}_p^{\mathrm{ha}}$

The objective of this section is to investigate the hyper-algebraic invariants of p-adic algebraic numbers that generate abelian extensions as well as tamely ramified extensions over $\mathbb{Q}_p$.

4.1. **Hyper-algebraic invariants of abelian extensions.** Let ζ_{p^n} be the p^n-th root of unity in Example 2.13, then it is easy to see that

	$\alpha = \zeta_p$	$\alpha = \zeta_{p^n}$ $(n \geq 2)$
$\mathfrak{F}_\alpha$	2	≥ 2
$\mathfrak{T}_\alpha$	$p-1$	$\geq p-1$

The following proposition gives a precise form of the above observations:

Proposition 4.1. *For any integer $n \geq 1$ and any p^n-th primitive root of unity ζ_{p^n}, we have $\mathfrak{T}_{\zeta_{p^n}} = p-1$ and*

$$\mathfrak{F}_{\zeta_{p^n}} \begin{cases} = \quad 2, & \text{if } n = 1, 2; \\ \text{divides } 2\cdot p^{n-2}, & \text{if } n \geq 3. \end{cases}$$

The key to prove this proposition is the following lemma:

Lemma 4.2. *Let $\alpha \in \mathbb{L}_p^{\mathrm{ha}}$ with $v_p(\alpha) = 0$. Then there exists a p-th root β of α in $\mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, p\cdot\mathfrak{F}_\alpha)$. In particular, if $C_{\frac{1}{p-1}}(\beta) = 0$, then β belongs to $\mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$.*

Proof. We apply the transfinite Newton algorithm on the equation $T^p - \alpha = 0$ to get a root β. Set $\beta = \sum_\omega [c_\omega]\cdot p^{k_\omega}$, where the ordinal ω run through the well-ordered set $\mathrm{supp}(\beta)$. Recall that for any ordinal ω, let $\beta_\omega = \sum_{\rho<\omega}[c_\rho]\cdot p^{k_\rho}$ and

$$\Phi_\omega(T) = (T+\beta_\omega)^p - \alpha = T^p + \sum_{k=1}^{p-1}\binom{p}{k}\beta_\omega^k \cdot T^{p-k} + \beta_\omega^p - \alpha.$$

The first step is easy: since $\beta_0 = 0$ and $\Phi_0(T) = T^p - \alpha$, the Newton polygon $\mathscr{N}\!ewt(\Phi_0)$ consists of a single horizontal segment with residue polynomial given by

$$\mathrm{Res}_{\Phi_0}(T) = T^p - C_0(\alpha) \in \mathbb{F}_{p^{\mathfrak{F}_\alpha}}[T],$$

which splits in $\mathbb{F}_{p^{\mathfrak{F}_\alpha}}$. This shows that $\beta_1 \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$ and $v_p(\beta_1) = 0$.

For any $\omega \geq 1$, since $v_p(\beta_\omega) = v_p(\beta_1) = 0$, we know that $v_p(\binom{p}{k}\beta_\omega^k) = 1$ for all $k = 1, 2, \cdots, p-1$. This implies that $\mathscr{N}ewt(\Phi_\omega)$ is determined by the point $(p, v_p(\beta_\omega^p - \alpha))$ for every $\omega \geq 1$.

Since $k_\omega \in \mathbb{Q}$ increases monotonically with respect to the ordinal ω, we set ω_0 to be the minimal ordinal ρ that satisfies $k_\rho \geq \frac{1}{p-1}$.

(1) Suppose $\omega < \omega_0$ and $\beta_\rho \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$ for every $\rho \leq \omega$. Then $\mathscr{N}ewt(\Phi_\omega)$ consists of a single segment with slope $k_\omega = s_{\max}^{\Phi_\omega} = \frac{1}{p} v_p(\beta_\omega^p - \alpha) < \frac{1}{p-1}$.

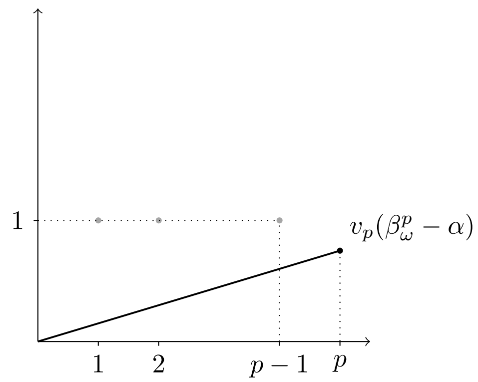

FIGURE 4.1. $\mathscr{N}ewt(\Phi_\omega)$, $1 \leq \omega < \omega_0$

Since $\beta_\omega^p - \alpha \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$ by Corollary 3.6, we know that

$$v_p(\beta_\omega^p - \alpha) \in \mathrm{supp}(\beta_\omega^p - \alpha) \subseteq \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}[1/p].$$

This implies that $k_\omega = \frac{1}{p} v_p(\beta_\omega^p - \alpha)$ also belongs to $\frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}[1/p]$. The residue polynomial of $\Phi_\omega(T)$ is given by

$$\mathrm{Res}_{\Phi_\omega}(T) = T^p + C_{v_p(\beta_\omega^p - \alpha)}(\beta_\omega^p - \alpha),$$

where $C_{v_p(\beta_\omega^p - \alpha)}(\beta_\omega^p - \alpha) \in \mathbb{F}_{p^{\mathfrak{F}_\alpha}}$. Thus any root of this residue polynomial lies in $\mathbb{F}_{p^{\mathfrak{F}_\alpha}}$. This shows that $\beta_{\omega+1} \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$. Since the case of limit ordinals is self-indicating, we can show by transfinite induction that $\beta_\omega \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$ for all $\omega \leq \omega_0$.

(2) Now we deal with $\omega = \omega_0 + 1$.

(a) If $k_{\omega_0} = s_{\max}^{\Phi_{\omega_0}} = \frac{1}{p-1}$, then $\mathscr{N}ewt(\Phi_{\omega_0})$ consists of a single segment with slope equals to

$$k_{\omega_0} = \frac{1}{p-1} = \frac{1}{p} v_p(\beta_{\omega_0}^p - \alpha) \in \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}[1/p].$$

Since this segment contains the point $(p-1, 1)$, one knows that

$$\mathrm{Res}_{\Phi_{\omega_0}}(T) = T^p + C_0(\beta_{\omega_0})^{p-1} T + C_{v_p(\beta_{\omega_0}^p - \alpha)}(\beta_{\omega_0}^p - \alpha) \in \mathbb{F}_{p^{\mathfrak{T}_\alpha}}[T],$$

whose root lies in $\mathbb{F}_{p^{p \cdot \mathfrak{F}_\alpha}}$. In this case, one has $\beta_{\omega_0+1} \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, p \cdot \mathfrak{F}_\alpha)$.

(b) If $k_{\omega_0} = s_{\max}^{\Phi_{\omega_0}} > \frac{1}{p-1}$, then $\mathscr{N}ewt(\Phi_{\omega_0})$ consists of two segments, where the vertexes of the segment with maximal slope is given by $(p-1, 1)$ and $(p, v_p(\beta_{\omega_0}^p - \alpha))$. Thus,

$$k_{\omega_0} = \frac{v_p(\beta_{\omega_0}^p - \alpha) - 1}{p - (p-1)} \in \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}[1/p]$$

and one has

$$\mathrm{Res}_{\Phi_{\omega_0}}(T) = C_0(\beta_{\omega_0})^{p-1}T + C_{v_p(\beta_{\omega_0}^p - \alpha)}(\beta_{\omega_0}^p - \alpha),$$

whose root lies in $\mathbb{F}_{p^{\mathfrak{F}_\alpha}}$. In this case, one has $\beta_{\omega_0+1} \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$.

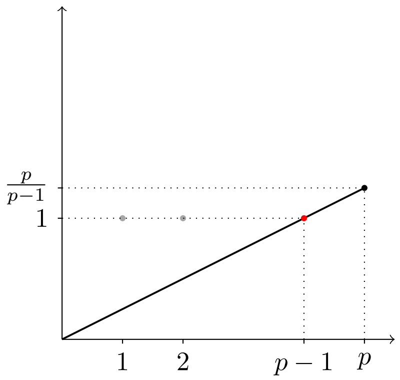

FIGURE 4.2. $\mathscr{N}\!ewt(\Phi_{\omega_0})$, if $k_{\omega_0} = \frac{1}{p-1}$

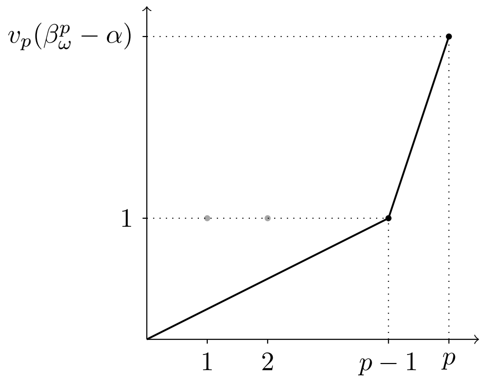

FIGURE 4.3. $\mathscr{N}\!ewt(\Phi_{\omega_0})$, if $k_{\omega_0} > \frac{1}{p-1}$

(3) For the case of $\omega > \omega_0$, we have $k_\omega > \frac{1}{p-1}$. With the same calculation as above, one can prove by transfinite induction that for any ordinal $\omega \geq \omega_0 + 1$, $\beta_\omega \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_{\beta_{\omega_0+1}})$.

The result follows. □

Additionally, we need the following auxiliary lemma:

Lemma 4.3. *For any p^2-th primitive root of unity ζ_{p^2}, there exists another p^2-th primitive root of unity ζ'_{p^2} and a p-th root of unity ξ_c (not necessarily primitive) that $\zeta_{p^2} = \zeta'_{p^2} \cdot \xi_c$ and $C_{\frac{1}{p-1}}(\zeta'_{p^2}) = 0$.*

Proof. Fix a $2(p-1)$-th primitive root of unity $\tilde{\zeta}_{2(p-1)}$. Let

$$\mathcal{W} := \left\{ \tilde{\zeta}_{2(p-1)}^{2k+1} : k \in \mathbb{N}_{<p-1} \right\} \subset \mathbb{F}_{p^2}.$$

By choosing $\zeta_{2(p-1)}$ in the expansion of the p^2-th primitive root of unity given by Example 2.13 (see also [WY21, Theorem 3.3]) in $\mathcal{W}$, we get $p-1$ different p^2-th primitive roots of unity $r_0, r_1, \cdots, r_{p-2}$, satisfying $C_{\frac{1}{p(p-1)}}(r_k) = \tilde{\zeta}_{2(p-1)}^{2k+1}$ and $C_{\frac{1}{p-1}}(r_k) = 0$ for every $k \in \mathbb{N}_{<p-1}$.

Similarly, for every $c \in \{0\} \cup \mathcal{W}$, there exists a p-th root of unity (not necessarily primitive) ξ_c that $v_p\Big(\xi_c - 1 - [c] \cdot p^{\frac{1}{p-1}}\Big) > \frac{1}{p-1}$. Thus for any $k \in \mathbb{N}_{<p-1}$ and $c \in \{0\} \cup \mathcal{W}$, $r_k \cdot \xi_c$ is a p^2-th primitive root of unity, satisfying $C_{\frac{1}{p(p-1)}}(r_k \cdot \xi_c) = \tilde{\zeta}_{2(p-1)}^{2k+1}$ and $C_{\frac{1}{p-1}}(r_k \cdot \xi_c) = c$. This enumerates all $p(p-1)$ p^2-th primitive roots of unity. The result follows. □

Proof of Proposition 4.1. The case of $n = 1$ follows immediately from [WY21, Proposition 3.4].

Let ζ_{p^2} be any p^2-th primitive root of unity. By Lemma 4.3, there exists another p^2-th primitive root of unity ζ'_{p^2} and a p-th root of unity ξ_c (not necessarily primitive) that $\zeta_{p^2}^p = \zeta'_{p^2} \cdot \xi_c$ and $C_{\frac{1}{p-1}}(\zeta'_{p^2}) = 0$. By applying Lemma 4.2, we have

$$\zeta'_{p^2} \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_{(\zeta'_{p^2})^p}, \mathfrak{F}_{(\zeta'_{p^2})^p}) = \mathbb{L}_p^{\mathrm{ha}}(p-1, 2).$$

Since $\xi_c \in \mathbb{L}_p^{\mathrm{ha}}(p-1,2)$, we know that $\zeta_{p^2} \in \mathbb{L}_p^{\mathrm{ha}}(p-1,2)$. On the other hand, by [WY21, Theorem 3.3], one has $\mathfrak{T}_{\zeta_{p^2}} \geq p-1$ and $\mathfrak{F}_{\zeta_{p^2}} \geq 2$. This implies that $\mathfrak{T}_{\zeta_{p^2}} = p-1$ and $\mathfrak{F}_{\zeta_{p^2}} = 2$.

When $n \geq 3$, we can set $\alpha = (\zeta_{p^n})^p$ in Lemma 4.2 inductively to get the result. One should notice that when $n \geq 3$, we no longer know if the analog of Lemma 4.3 holds for ζ_{p^n}. Thus the hyper-inertia index is multiplied by p when n increases by 1. □

Corollary 4.4. *For any positive integer $m = r \cdot p^{v_p(m)}$ with $\gcd(r,p) = 1$ and any m-th primitive root of unity ζ_m, one has*

(1) If $v_p(m) = 0$, then $\mathfrak{T}_{\zeta_m} = 1$ and $\mathfrak{F}_{\zeta_m} = \mathrm{ord}_r\, p$.
(2) If $v_p(m) \geq 1$, then $\mathfrak{T}_{\zeta_m} \mid p-1$ and

$$\mathfrak{F}_{\zeta_m} \mid \begin{cases} \mathrm{lcm}(2, \mathrm{ord}_r\, p), & \text{if } v_p(m) = 1, 2; \\ \mathrm{lcm}\big(2 \cdot p^{v_p(m)-1}, \mathrm{ord}_r\, p\big), & \text{if } v_p(m) \geq 3. \end{cases}$$

Proof. It suffices to note that any r-th root of unity lies in $\mathbb{F}_{p^{\mathrm{ord}_r p}}$. □

With the power of the local Kronecker-Weber theorem, we can generalize this result to those p-adic algebraic numbers that generate abelian extensions over $\mathbb{Q}_p$:

Theorem 4.5. *Let $\alpha \in \overline{\mathbb{Q}}_p$ be a p-adic algebraic number with $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ an abelian extension of degree n. Denote by $\mathbf{f}_{\mathbb{Q}_p(\alpha)}$ the local conductor of $\mathbb{Q}_p(\alpha)$ over $\mathbb{Q}_p$. Then*

(1) If $\mathbf{f}_{\mathbb{Q}_p(\alpha)} = 0$, then $\mathfrak{T}_\alpha = 1$ and $\mathfrak{F}_\alpha = n$.
(2) If $\mathbf{f}_{\mathbb{Q}_p(\alpha)} \geq 1$, then $\mathfrak{T}_\alpha \mid p-1$ and

$$\mathfrak{F}_\alpha \mid \begin{cases} \mathrm{lcm}(2, n), & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} = 1, 2; \\ \mathrm{lcm}\Big(2 \cdot p^{\mathbf{f}_{\mathbb{Q}_p(\alpha)}-1}, n\Big), & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} \geq 3. \end{cases}.$$

To prove this theorem, the following effective form of the local Kronecker-Weber theorem is needed:

Lemma 4.6. *Let $K/\mathbb{Q}_p$ be an abelian extension of degree n with conductor $\mathbf{f}_K$ and let $m = (p^n - 1)p^{\mathbf{f}_K}$. Then $K \subseteq \mathbb{Q}_p(\zeta_m)$.*

Proof. By [Gui18, Lemma 4.11] and its proof, there exists $s \geq 1$ that

$$\langle p^s \rangle \times U_{\mathbb{Q}_p}^{(\mathbf{f}_K)} \subseteq \mathcal{N}_{K/\mathbb{Q}_p} K^\times.$$

It follows that $K \subseteq \mathbb{Q}_p\Big(\zeta_{(p^s-1)p^{\mathbf{f}_K}}\Big)$ by the proof of [Gui18, Theorem 13.27]. On the other hand, we have $K \subseteq \mathbb{Q}_p\Big(\zeta_{(p^n-1)p^{v_p(n)+2}}\Big)$ by [KS22, Theorem 3.1]. Since

$$\mathbb{Q}_p\Big(\zeta_{(p^s-1)p^{\mathbf{f}_K}}\Big) \cap \mathbb{Q}_p\Big(\zeta_{(p^n-1)p^{v_p(n)+2}}\Big) \subseteq \mathbb{Q}_p(\zeta_m),$$

we have $K \subseteq \mathbb{Q}_p(\zeta_m)$. □

Proof of Theorem 4.5. Let $m = (p^n - 1)p^{\mathbf{f}_{\mathbb{Q}_p(\alpha)}}$. By Lemma 4.6, we know that $\alpha \in \mathbb{Q}_p(\zeta_m)$.

Note $\mathrm{ord}_{p^n-1}\, p = n$. By Corollary 4.4, we know that

$$\mathfrak{T}_{\zeta_m} = \begin{cases} 1, & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} = 0; \\ p-1, & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} \geq 1. \end{cases}$$

and

$$\mathfrak{F}_{\zeta_m} \begin{cases} = & n, & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} = 0; \\ = & \operatorname{lcm}(2,n), & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} = 1,2; \\ \text{divides} & \operatorname{lcm}\Big(2 \cdot p^{\mathbf{f}_{\mathbb{Q}_p(\alpha)}-1}, n\Big), & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} \geq 3. \end{cases}$$

Since $\alpha \in \mathbb{Q}_p(\zeta_m) \subseteq \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_{\zeta_m}, \mathfrak{F}_{\zeta_m})$, the result follows. □

4.2. **Criterion for tamely ramified extensions.**

Theorem 4.7. *Let $\alpha \in \mathbb{L}_p^{\mathrm{ha}}$ be a hyper-algebraic element in $\mathbb{L}_p$. Then $\mathbb{Q}_p(\alpha)$ is tamely ramified over $\mathbb{Q}_p$ if and only if* $\operatorname{supp}(\alpha) \subseteq \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}$. *In this situation, we have $\mathfrak{T}_\alpha = \mathfrak{e}_\alpha^{\mathrm{t}}$, $\mathfrak{f}_\alpha \mid \mathfrak{F}_\alpha$ and $\mathfrak{F}_\alpha \mid \operatorname{ord}_{\mathfrak{e}_\alpha^{\mathrm{t}}(p^{\mathfrak{f}_\alpha}-1)} p$.*

The proof of this theorem relies on the following lemma:

Lemma 4.8. *Let $\alpha \in \overline{\mathbb{Q}}_p$ be a p-adic algebraic number with $\mathbb{Q}_p(\alpha)$ tamely ramified over $\mathbb{Q}_p$. Then there exist an element $\xi \in \mathbb{F}_{p^c}$, with*

$$c = \operatorname{ord}_{\mathfrak{e}_\alpha^{\mathrm{t}}(p^{\mathfrak{f}_\alpha}-1)} p \leq \mathfrak{f}_\alpha \cdot \mathfrak{e}_\alpha^{\mathrm{t}},$$

that

$$\mathbb{Q}_p(\alpha) = \mathbb{Q}_p\Big([\xi] \cdot p^{\frac{1}{\mathfrak{e}_\alpha^{\mathrm{t}}}}\Big).$$

Proof. The proof is nothing but a slight improvement of [htt].

Let $\mathcal{O}_K$ be the ring of integer of $K := \mathbb{Q}_p(\alpha)$ with a uniformizer π_K. Since $K/\mathbb{Q}_p$ is tamely ramified, there exists a unit u in $\mathcal{O}_K^\times$ that $\pi_K^{\mathfrak{e}_\alpha^{\mathrm{t}}} = p \cdot u$. By the structure theorem of CDVR, one may write $u \in [\zeta] + \pi_K\mathcal{O}_K$, with $\zeta \in \mathbb{F}_{p^{\mathfrak{f}_\alpha}}$.

Since $\gcd(\mathfrak{e}_\alpha^{\mathrm{t}}, p) = 1$ and $u^{-1}[\zeta] - 1 \in \pi_K\mathcal{O}_K$, the series

$$u^{-1/\mathfrak{e}_\alpha^{\mathrm{t}}}[\zeta]^{1/\mathfrak{e}_\alpha^{\mathrm{t}}} = (1 + (u^{-1}[\zeta]-1))^{1/\mathfrak{e}_\alpha^{\mathrm{t}}} = \sum_{k=0}^{\infty} \binom{1/\mathfrak{e}_\alpha^{\mathrm{t}}}{k} (u^{-1}[\zeta]-1)^k$$

converges in $\mathcal{O}_K^\times$. Thus the element

$$\pi_K \cdot u^{-1/\mathfrak{e}_\alpha^{\mathrm{t}}}[\zeta]^{1/\mathfrak{e}_\alpha^{\mathrm{t}}} = p^{1/\mathfrak{e}_\alpha^{\mathrm{t}}} \cdot [\zeta^{1/\mathfrak{e}_\alpha^{\mathrm{t}}}]$$

is also a uniformizer of $\mathcal{O}_K$. Take $\xi := \zeta^{1/\mathfrak{e}_\alpha^{\mathrm{t}}}$, then ξ is a root of the polynomial $f(T) = T^{\mathfrak{e}_\alpha^{\mathrm{t}}(p^{\mathfrak{f}_\alpha}-1)} - 1 \in \mathbb{F}_p[T]$, which belongs to the splitting field $\mathbb{F}_{p^c}$ of $f(T)$. Since the degree of the minimal polynomial of ξ over $\mathbb{F}_{p^{\mathfrak{f}_\alpha}}$ is at most $\mathfrak{e}_\alpha^{\mathrm{t}}$, one knows that $\operatorname{ord}_{\mathfrak{e}_\alpha^{\mathrm{t}}(p^{\mathfrak{f}_\alpha}-1)} p \leq \mathfrak{f}_\alpha \cdot \mathfrak{e}_\alpha^{\mathrm{t}}$. □

Proof of Theorem 4.7. If $\operatorname{supp}(\alpha) \subseteq \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}$, we can write $\alpha = \sum_{k \gg -\infty}^{+\infty} [r_k] \cdot p^{\frac{k}{\mathfrak{T}_\alpha}}$, where $r_k \in \mathbb{F}_{p^{\mathfrak{F}_\alpha}}$ for all k. Thus α lies in $\mathbb{Q}_{p^{\mathfrak{F}_\alpha}}\Big(p^{\frac{1}{\mathfrak{T}_\alpha}}\Big)$, where $\mathbb{Q}_{p^{\mathfrak{F}_\alpha}} := \operatorname{Frac} W(\mathbb{F}_{p^{\mathfrak{F}_\alpha}})$ is the unique unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_{p^{\mathfrak{F}_\alpha}}$. This shows that $\mathbb{Q}_p(\alpha)$ is tamely ramified over $\mathbb{Q}_p$.

Conversely, if $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ is tamely ramified, then we have $\mathbb{Q}_p(\alpha) = \mathbb{Q}_p\Big([\xi] \cdot p^{\frac{1}{\mathfrak{e}_\alpha^{\mathrm{t}}}}\Big)$ by Lemma 4.8, where $\xi \in \mathbb{F}_{p^c}$ with $c = \operatorname{ord}_{\mathfrak{e}_\alpha^{\mathrm{t}}(p^{\mathfrak{f}_\alpha}-1)} p$. This shows that $\operatorname{supp}(\alpha) \subseteq \frac{1}{\mathfrak{e}_\alpha^{\mathrm{t}}}\mathbb{Z}$, implying the elements in $\operatorname{supp}(\alpha)$ has non-negative p-adic valuation. Thus $\operatorname{supp}(\alpha) \subseteq \mathbb{Z}_{(p)} \cap \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}[1/p] = \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}$.

Notice that the inclusion $\alpha \in \mathbb{Q}_{p^{\mathfrak{F}_\alpha}}\Big(p^{\frac{1}{\mathfrak{T}_\alpha}}\Big)$ implies $\mathfrak{e}_\alpha^{\mathrm{t}} \mid \mathfrak{T}_\alpha$ and $\mathfrak{f}_\alpha \mid \mathfrak{F}_\alpha$. On the other hand, the equality $\mathbb{Q}_p(\alpha) = \mathbb{Q}_p\Big([\xi] \cdot p^{\frac{1}{\mathfrak{e}_\alpha^{\mathrm{t}}}}\Big)$ gives us the inclusion $\alpha \in \mathbb{Q}_p\Big([\xi] \cdot p^{\frac{1}{\mathfrak{e}_\alpha^{\mathrm{t}}}}\Big) \subset \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{e}_\alpha^{\mathrm{t}}, c)$, showing that $\mathfrak{T}_\alpha \mid \mathfrak{e}_\alpha^{\mathrm{t}}$, $\mathfrak{F}_\alpha \mid c$ and $\mathfrak{e}_\alpha^{\mathrm{t}} \mid \mathfrak{T}_\alpha$. □

4.3. **Heuristic discussion for general extensions.** We have seen in Proposition 3.7 and Theorem 4.5 that given a p-adic algebraic number α, its hyper-algebraic invariants $\mathfrak{T}_\alpha$ and $\mathfrak{F}_\alpha$ are closely related to its arithmetic invariants, i.e. $[\mathbb{Q}_p(\alpha)\colon \mathbb{Q}_p]$, $\mathfrak{e}^{\mathrm{t}}_\alpha$ and $\mathbf{f}_{\mathbb{Q}_p(\alpha)}$. These invariants can be determined by its minimal polynomial over $\mathbb{Q}_p$. However, the minimal polynomial is not enough to determine the exact value of $\mathfrak{T}_\alpha$ and $\mathfrak{F}_\alpha$ in general. For example, the elements $\alpha_1 = p^{1/p}$ and $\alpha_2 = p^{1/p}\cdot\zeta_p$ shares the same minimal polynomial $T^p - p$ over $\mathbb{Q}_p$ but $\mathfrak{T}_{\alpha_1} = \mathfrak{F}_{\alpha_1} = 1$ while $\mathfrak{T}_{\alpha_2} = p-1$ and $\mathfrak{F}_{\alpha_2} = 2$ by Proposition 4.1.

A small-scale numerical experiment indicates the following heuristic patterns:

(1) The hyper-inertia index $\mathfrak{F}_\alpha$ always divides $[\mathbb{Q}_p(\alpha)\colon \mathbb{Q}_p]$.

(2) If $f(T) \in \mathbb{Q}_p[T]$ is irreducible, denote by $\mathfrak{e}^{\mathrm{t}}_f$ the tame ramification index of $\mathbb{Q}_p[T]/f(T)$ over $\mathbb{Q}_p$. Then for any root α of f, $\mathfrak{T}_\alpha$ always divides $\mathfrak{e}^{\mathrm{t}}_f$. There exists at least one root β of f that $\mathfrak{T}_\beta = \mathfrak{e}^{\mathrm{t}}_f$.

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