HYPER-ALGEBRAIC INVARIANTS OF p-ADIC ALGEBRAIC NUMBERS

SHANWEN WANG AND YIJUN YUAN

Abstract. Let $p \geq 3$ be a prime. The hyper-algebraic elements in the p-adic Mal'cev-Neumann field $\mathbb{L}_p$ form an algebraically closed subfield $\mathbb{L}_p^{\mathrm{ha}}$. In this article, we clarify the relations among the fields $\mathbb{L}_p^{\mathrm{ha}}$, $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$. We introduce two arithmetic invariants (hyper-tame index and hyper-inertia index) of hyper-algebraic elements and study the relation between these invariants and classical arithmetic invariants of p-adic algebraic numbers. Finally, we give a criterion for hyper-algebraic elements to be tamely ramified over $\mathbb{Q}_p$.

Contents

1. Introduction

Let $p \geq 3$ be a prime throughout this article. The p-adic Mal'cev-Neumann field $\mathbb{L}_p := W(\overline{\mathbb{F}}_p)((p^{\mathbb{Q}}))$, constructed in [Poo93], is the unique minimal spherically complete extension of the field $\mathbb{C}_p$ of p-adic complex numbers. An element $f \in \mathbb{L}_p$ can be written uniquely in the form

$$f = \sum_{q \in \mathbb{Q}} [r_q] p^q, \text{ where } [\cdot] \colon \overline{\mathbb{F}}_p \longrightarrow W(\overline{\mathbb{F}}_p) \text{ is the Teichmüller character}$$

and $\mathrm{supp}(f) = \{q \in \mathbb{Q} \colon r_q \neq 0\}$ a well-ordered subset of $\mathbb{Q}$. Thus, an element $f = \sum_{q \in \mathbb{Q}} [r_q] p^q$ of $\mathbb{L}_p$ is completely determined by its support $\mathrm{supp}(f)$ and the set $\{r_q\}_{q \in \mathbb{Q}}$ of its coefficients.

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The spherically complete condition is crucial in non-Archimedean functional analysis (see [Sch02, Proposition 9.2] for a concrete example). In arithmetic geometry, it also serves as an intermediate hypothesis in Scholze and Weinstein's classification of p-divisible groups over the ring $\mathcal{O}_{\mathbb{C}_p}$ of integers of $\mathbb{C}_p$ (cf. [SW13, Proposition 5.2.5]). Besides the importance of spherical completeness, it is surprising that not much arithmetic of $\mathbb{L}_p$ is investigated. We summarize several results from the literature:

(1) In [Lam86], Lampert introduced the notion of p-adic Mal'cev-Neumann series. In particular, he proved that the elements in $\mathbb{L}_p$, satisfying that the accumulating points of the support are all rational, form an algebraically closed field (cf. [Lam86, Theorem 2]).
(2) In [Poo93], Poonen gave a rigorous construction of the field $\mathbb{L}_p$ and systematically studied various aspects of this field. In particular, a necessary condition for an element of $\mathbb{L}_p$ to be algebraic over $\mathbb{Q}_p$, which is claimed by Lampert in [Lam86, p. 282], is proved in [Poo93, Corollary 8].
(3) Based on an idea of Lampert, Kedlaya proposed a transfinite Newton algorithm (cf. [Ked01, Proposition 1]) to prove the algebraic closeness of $\mathbb{L}_p$ effectively, which is extracted in [WY21, Algorithm 1].
(4) In [Ked17, Theorem 13.4], Kedlaya gave a necessary and sufficient condition for an element in $\mathbb{L}_p$ to be a p-adic complex number, in terms of the so-called "p-quasi-automatic elements".
(5) The truncated expansions of roots of unity in $\mathbb{L}_p$ are studied in [WY21, Theorem 3.3] and [WY23, Theorem 1.6]. Based on these results, the uniformizers of the p-adic false Tate curve extensions $\mathbb{K}_p^{m,n} := \mathbb{Q}_p\big(\zeta_{p^m}, p^{1/p^n}\big)$ for $(m,n) \in (\{2\} \times \mathbb{Z}_{\geq 1}) \cup (\mathbb{Z}_{\geq 3} \times \{1\})$ are constructed (cf. [WY21; WY24]).
(6) On the field $\mathbb{L}_p$, we can define a canonical Frobenius map by the formula
$$\varphi\colon \sum_{q\in\mathbb{Q}} [r_q] p^q \longmapsto \sum_{q\in\mathbb{Q}} [r_q^p] p^q.$$
In [Efi24], Efimov proved that φ acts on the systems of p^n-th roots of unity by taking inverse. Note that one can view the complex conjugation as the Frobenius automorphism of $\mathbb{C}$, and the result of Efimov justifies that the Frobenius φ can be viewed as the complex conjugation on $\mathbb{L}_p$.

The purpose of this article is to answer several natural questions concerning the arithmetic of the field $\mathbb{L}_p$, which we make precise in the following.

1.1. **Criterion of algebraicity.** By [Lam86, p. 282] and [Poo93, Corollary 8], if $f \in \mathbb{L}_p$ is algebraic over $\mathbb{Q}_p$, then it satisfies the following conditions:

(1) there exists a positive integer N such that $\mathrm{supp}(f) \subseteq \frac{1}{N}\mathbb{Z}[1/p]$;
(2) there exists a positive integer k such that $r_q \in \mathbb{F}_{p^k}$ for all $q \in \mathrm{supp}(f)$.

An element $f \in \mathbb{L}_p$ satisfying the above conditions is called *hyper-algebraic.* The set $\mathbb{L}_p^{\mathrm{ha}}$ of hyper-algebraic elements in $\mathbb{L}_p$ forms an algebraically closed field containing $\mathbb{Q}_p$. As a result, all p-adic algebraic numbers are hyper-algebraic, i.e. $\overline{\mathbb{Q}}_p \subseteq \mathbb{L}_p^{\mathrm{ha}}$. Our first result is a clarification of relations among the fields $\mathbb{L}_p^{\mathrm{ha}}$, $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$:

Theorem A (cf. Theorem 3.3). *The field $\mathbb{L}_p^{\mathrm{ha}}$ is strictly larger than $\overline{\mathbb{Q}}_p$, and it is neither complete nor a subfield of $\mathbb{C}_p$.*

For a hyper-algebraic element $\alpha \in \mathbb{L}_p^{\mathrm{ha}}$, we introduce two new invariants of α, called the hyper-tame index $\mathfrak{T}_\alpha$ and hyper-inertia index $\mathfrak{F}_\alpha$, defined to be the minimal integers N and k in the conditions given by Poonen respectively. For a p-adic algebraic number $\alpha \in \overline{\mathbb{Q}}_p$, its hyper-algebraic invariants $\mathfrak{T}_\alpha$ and $\mathfrak{F}_\alpha$ are closely related to its usual arithmetic invariants.

Theorem B (cf. Theorem 4.1, Theorem 4.9)**.** *Let α be a p-adic algebraic number.*

(1) The hyper-algebraic invariants $\mathfrak{T}_\alpha$ and $\mathfrak{F}_\alpha$ do not exceed $[\mathbb{Q}_p(\alpha)\colon \mathbb{Q}_p]$;

(2) Suppose $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ is an abelian extension of degree n. Denote by $\mathbf{f}_{\mathbb{Q}_p(\alpha)}$ the local conductor of $\mathbb{Q}_p(\alpha)$ over $\mathbb{Q}_p$. Then

(a) If $\mathbf{f}_{\mathbb{Q}_p(\alpha)} = 0$, then $\mathfrak{T}_\alpha = 1$ and $\mathfrak{F}_\alpha = n$.

(b) If $\mathbf{f}_{\mathbb{Q}_p(\alpha)} \geq 1$, then $\mathfrak{T}_\alpha \mid p-1$ and

$$\mathfrak{F}_\alpha \mid \begin{cases} \operatorname{lcm}(2,n), & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} = 1,2; \\ \operatorname{lcm}\Big(2\cdot p^{\mathbf{f}_{\mathbb{Q}_p(\alpha)}-1}, n\Big), & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} \geq 3. \end{cases}.$$

Remark 1.1. *The proof of this result is based on our computation of the truncated expansion of ζ_{p^n} (cf. [WY21; WY23], and also see Example 2.13 for the precise formula).*

Remark 1.2. *For $\alpha \in \mathbb{L}_p$, we denote by $[C_{\frac{1}{p-1}}(\alpha)]$ the coefficient of index $\frac{1}{p-1}$ of the canonical expansion of α. Based on the truncated expansion of ζ_{p^n} (cf. Example 2.13), we conjecture that for any integer $n \geq 2$ and p^n-th primitive root of unity ζ_{p^n}, there exists another p^n-th primitive root of unity ζ'_{p^n} with $C_{\frac{1}{p-1}}(\alpha) = 0$ such that $\zeta_{p^n}^{p^{n-1}} = \left(\zeta'_{p^n}\right)^{p^{n-1}}$.*

If this conjecture holds[1]*, then $\mathfrak{F}_{\zeta_{p^n}} = 2$ for every $n \geq 2$, and consequently $\mathfrak{F}_\alpha$ divides* $\operatorname{lcm}(2,n)$ *for all ramified cases in the above theorem. See the proof of Proposition 4.5 for more details. Note that this conjecture is true when $n = 2$ (cf. Lemma 4.7).*

Our third result is to give a criterion for hyper-algebraic element to be tamely ramified over $\mathbb{Q}_p$:

Theorem C (cf. Theorem 4.11)**.** *Let $\alpha \in \mathbb{L}_p^{\mathrm{ha}}$ be a hyper-algebraic element in $\mathbb{L}_p$. Then $\mathbb{Q}_p(\alpha)$ is tamely ramified over $\mathbb{Q}_p$ if and only if* $\operatorname{supp}(\alpha) \subseteq \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}$. *In this situation, we have $\mathfrak{T}_\alpha = \mathfrak{e}_\alpha$, $\mathfrak{f}_\alpha \mid \mathfrak{F}_\alpha$ and $\mathfrak{F}_\alpha \mid c$, where $c := \operatorname{ord}_{\operatorname{lcm}(\mathfrak{e}_\alpha, p^{\mathfrak{f}_\alpha}-1)} p$ and $\mathfrak{f}_\alpha$ (resp. $\mathfrak{e}_\alpha$) is the inertia degree (resp. the ramification index) of the extension $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$.*

Remark 1.3. *It seems that our method for abelian and tamely ramified extensions can hardly be generalized to general extensions. For these two special cases, the key ingredient is to find an extension K over $\mathbb{Q}_p(\alpha)$, which is generated by certain more "controllable" elements. In the abelian case, we use the cyclotomic extension by the local Kronecker-Weber theorem while in the tamely ramified case, we used the radical extension by Lemma 4.12. However, in general, we don't know how to find such a more "controllable" field.*

1.2. **Distinguishing roots of irreducible polynomial over $\mathbb{Q}_p$.** The canonical expansion of an element in $\mathbb{L}_p$ is fairly an analogy of the polar coordinate of a complex number. In fact, the support $\operatorname{supp}(f)$ of $f \in \mathbb{L}_p$ corresponds to the modulus of a complex number while the set $\{r_q\}_{q\in\mathbb{Q}}$ of coefficients of the expansion of f corresponds to the argument of a complex number. As a result, such an expansion can be used to make a distinction of roots of polynomials over $\mathbb{Q}_p$.

Given a p-adic algebraic number α, the usual arithmetic invariants (i.e. the degree, ramification index and inertia degree of the extension $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$) of α are determined by its minimal polynomial over $\mathbb{Q}_p$. Thus, the usual arithmetic invariants can not be used to distinguish the conjugates of α under the action of absolute

[1]We notice that in a recent preprint (cf. [Efi24]), Efimov claimed (ibid., Section 2) that his main theorem (ibid.) implies $\mathfrak{F}_{\zeta_{p^n}} = 2$ for every $n \geq 1$. With his result, we can bypass the aforementioned conjecture.

Galois group of $\overline{\mathbb{Q}}_p$. We observe that in general the minimal polynomial of α over $\mathbb{Q}_p$ is insufficient to determine the exact value of $\mathfrak{T}_\alpha$ and $\mathfrak{F}_\alpha$. For example, the elements $\alpha_1 = p^{1/p}$ and $\alpha_2 = p^{1/p} \cdot \zeta_p$ shares the same minimal polynomial $T^p - p$ over $\mathbb{Q}_p$ but $\mathfrak{T}_{\alpha_1} = \mathfrak{F}_{\alpha_1} = 1$ while $\mathfrak{T}_{\alpha_2} = p - 1$ and $\mathfrak{F}_{\alpha_2} = 2$ by Proposition 4.5. Thus, it provides the possibility to make a distinction of root of a polynomial using these two new invariants.

On the other hand, for a p-adic algebraic number α, its classical arithmetic invariants are related to the hyper-algebraic invariants of all its conjugates. The above example suggests that it makes sense to consider the hyper-algebraic invariants of all conjugate of α at the same time. Let $\mathfrak{T}(\alpha)$ (resp. $\mathfrak{F}(\alpha)$) be the set of hyper-tame indices (resp. hyper-inertia indices) of all the conjugates of α, equipped with the partial order defined by divisibility. A small-scale numerical experiment indicates the following heuristic patterns:

(1) The degree of the minimal polynomial of α over $\mathbb{Q}_p$ is always an upper bound of $\mathfrak{F}(\alpha)$ in $\mathbb{Z}_{>0}$ with respect to the order defined by divisibility.
(2) The p-power-free part of the ramification index of the field $\mathbb{Q}_p(\alpha)$ over $\mathbb{Q}_p$ is always the unique minimal element in $\mathfrak{T}(\alpha)$.

1.3. **Related works.** We mention some potential approaches to study the canonical expansion of general p-adic algebraic numbers in $\mathbb{L}_p^{\mathrm{ha}}$:

(1) In [Ked17, Theorem 13.4], Kedlaya gives a characterization of the canonical expansion of elements of $\mathcal{O}_{\mathbb{C}_p}$ in $\mathbb{L}_p$ in terms of the so-called "p-quasi-automatic elements". Extracting additional arithmetic information from these logic-derived objects could offer a fresh perspective on comprehending the hyper-algebraic invariants.
(2) In [Lis23], Lisinski uses a variant of Newton algorithm to give an upper bound of the order type of $\mathrm{supp}(\alpha)$ for element α in $\overline{\mathbb{F}_p((t))} \subset \overline{\mathbb{F}}_p((t^{\mathbb{Q}}))$. Besides that, Lisinski also designs an algorithm to give upper bounds for the characteristic p analog of hyper-algebraic invariants for elements in $\overline{\mathbb{F}_p((t))} \subset \overline{\mathbb{F}}_p((t^{\mathbb{Q}}))$. It is possible to develop a mixed-characteristic analog of Lisinski's results for $\overline{\mathbb{Q}}_p \subset \mathbb{L}_p$ and to compare with Theorem 4.1 of this paper.
(3) Inspired by the pioneering work [Don+24] of Dong-He-Jin-Schremmer-Yu, which using machine learning approach to study the geometry of affine Deligne-Lusztig varieties, we wonder if the machine learning method can help to identify hidden structures in the canonical expansion of a p-adic algebraic number in $\mathbb{L}_p^{\mathrm{ha}}$.

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2. Preliminaries on valued fields

2.1. **Maximally complete fields and Mal'cev-Neumann fields.** The main objective of this subsection is to justify the notion of immediate maximally complete extension of a valued field, in particular, of the field $\mathbb{C}_p$ of p-adic complex numbers.

Definition 2.1. *Let (F, v) be a valued field.*

(1) *Say (E, w) is an* ***immediate extension*** *of F if it is an extension of (F, v) and has the same value group and residue field as F.*
(2) *Say (F, v) is* ***maximally complete*** *if it has no proper immediate extension.*

Unsurprisingly, one has the following result

Proposition 2.2 ([Poo93, Proposition 6])**.**

(1) *Maximally complete fields are complete.*
(2) *If a maximally complete field has divisible value group and algebraically closed residue field, then itself is algebraically closed.*

Remark 2.3.

(1) *The proof of this Proposition, which is due to MacLane, is not effective, i.e. it does not give an algorithm to construct a root of a given polynomial over F.*
(2) *Kaplansky showed in [Kap42, Section 5] that there exist valued fields with two immediate maximally complete extensions that are not isomorphic as fields.*

Definition 2.4. *Let F be a valued field and (E_1, w_1), (E_2, w_2) be two extension of F.*

(1) *Say E_1 and E_2 are* ***analytically equivalent*** *if there exists a F-isomorphism of field $i: E_1 \longrightarrow E_2$ such that $w_2(i(x)) = w_1(x)$ for any $x \in E_1$.*
(2) *Say E_1 embeds into E_2 if E_1 is analytically equivalent to a subfield of E_2.*

Theorem 2.5 ([Poo93, Corollary 6])**.** *Every valued field F has an immediate maximally complete extension. If F has divisible value group and algebraically closed residue field, then the immediate maximally complete extension is unique up to analytic equivalence.*

A standard way to produce maximally complete fields is to consider the Mal'cev-Neumann fields which we recall in the rest of this paragraph.

Definition 2.6 ([Poo93, Section 3])**.** *Let R be a commutative ring and G be an ordered group.*

(1) *For any $f \in \mathrm{Hom}_{\mathrm{Set}}(G, R)$, we define the* ***support*** *of f to be*
$$\mathrm{supp}(f) = \{g \in G \colon f(g) \neq 0\}.$$
(2) *Define the set of* ***Mal'cev-Neumann series*** *over R with value group G to be*
$$R(\!(G)\!) := \{f \in \mathrm{Hom}_{\mathrm{Set}}(G, R) \colon \mathrm{supp}(f) \text{ is well-ordered}\}.$$
By introducing a formal variable t, elements in $R(\!(G)\!)$ will also be written as $\sum_{g\in G} r_g t^g$, where $r_g \in R$ for all $g \in G$.

Proposition 2.7 ([Poo93, Lemma 1, Corollary 2])**.** *Let R be a commutative ring and G be an ordered group.*

(1) *With identity $1 \cdot t^0$ and addition as well as multiplication given by*
$$\sum_{g\in G} b_g t^g + \sum_{g\in G} b_g t^g := \sum_{g\in G} (a_g + b_g) t^g, \ \sum_{g\in G} b_g t^g \cdot \sum_{g\in G} b_g t^g := \sum_{g\in G} \left(\sum_{h\in G} a_h b_{g-h} \right) t^{,}$$
$R(\!(G)\!)$ forms a commutative ring.

(2) *If R is a field, then so does $R(\!(G)\!)$. Moreover, with the map*

$$v\colon R(\!(G)\!) \longrightarrow G \cup \{\infty\},\ f \longmapsto \begin{cases} \min \operatorname{supp}(f), & \text{if } f \neq 0 \\ \infty, & \text{if } f = 0 \end{cases},$$

$R(\!(G)\!)$ becomes a valued field with value group G and residue field R.

Note that $\operatorname{char} R(\!(G)\!) = \operatorname{char} R$, we call $R(\!(G)\!)$ the **equal-characteristic Mal'cev-Neumann field** over R with value group G, also denoted as $R(\!(t^G)\!)$ with respect to the formal variable t.

Theorem 2.8 ([Poo93, Proposition 3, Corollary 3, Proposition 5])**.** *Let k be a perfect field of characteristic p and G be an ordered group containing $\mathbb{Z}$ as a subgroup. Besides that, let*

$$\mathcal{N} := \left\{ \sum_{g\in G} r_g t^g \in W(k)(\!(t^G)\!)\colon \text{ for every } g \in G,\ \sum_{n\in\mathbb{Z}} r_{g+n} p^n = 0 \right\},$$

where $W(k)$ is the ring of Witt vectors of k. Then

(1) *$\mathcal{N}$ is a maximal ideal of $W(k)(\!(t^G)\!)$, which makes $W(k)(\!(p^G)\!) := W(k)(\!(t^G)\!)/\mathcal{N}$ a field[2], called the **p-adic Mal'cev-Neumann field**.*

(2) *Every element in $W(k)(\!(p^G)\!)$ can be uniquely (and formally) written as*

$$\sum_{g\in G} [r_g] p^g,$$

where $r_g \in k$ for all $g \in G$ and $[\cdot]\colon k \longrightarrow W(k)$ is the Teichmüller lift.

(3) *For $f = \sum_{g\in G} [r_g] p^g$, define the **support** of f to be*

$$\operatorname{supp}(f) = \{g \in G\colon r_g \neq 0\}.$$

Then the map

$$v\colon W(k)(\!(G)\!)/\mathcal{N} \longrightarrow G \cup \{\infty\},\ f \mapsto \begin{cases} \min \operatorname{supp}(f), & \text{if } f \neq 0 \\ \infty, & \text{if } f = 0 \end{cases}$$

makes $W(k)(\!(G)\!)/\mathcal{N}$ a mixed-characteristic valued field with value group G and residue field k.

Theorem 2.9 ([Poo93, Theorem 1])**.** *The equal-characteristic and p-adic Mal'cev-Neumann fields are maximally complete.*

Theorem 2.10 ([Poo93, Corollary 5, Corollary 6])**.** *Let F be a valued field with value group G and residue field k with $\operatorname{char} k = 0$ or p. Let $\widetilde{G}$ be a divisible group that contains G.*

(1) *The field F embeds into the Mal'cev-Neumann field*

$$\begin{cases} k^{\text{alg}}(\!(t^{\widetilde{G}})\!), & \text{if } \operatorname{char} F = \operatorname{char} k; \\ W(k^{\text{alg}})(\!(p^{\widetilde{G}})\!), & \text{if } \operatorname{char} F \neq \operatorname{char} k; \end{cases}$$

where k^{alg} is an algebraic closure of k.

(2) *If $G = \widetilde{G}$ and $k = k^{\text{alg}}$, then the Mal'cev-Neumann field*

$$\begin{cases} k(\!(t^G)\!), & \text{if } \operatorname{char} F = \operatorname{char} k; \\ W(k)(\!(p^G)\!), & \text{if } \operatorname{char} F \neq \operatorname{char} k; \end{cases}$$

is the unique (up to analytic equivalence) immediate maximally complete extension of F (cf. Theorem 2.5).

[2] Intuitively speaking, $W(k)(\!(p^G)\!)$ is obtained by replacing the formal variable t of elements in $W(k)(\!(t^G)\!)$ by the prime p.

Example 2.11. *It is well-known that $\mathbb{C}_p$ is not maximally complete (cf. [BS18, Theorem 4.8, Theorem 6.7]). Since it has value group $\mathbb{Q}$ and residue field $\overline{\mathbb{F}}_p$, we can apply Theorem 2.10 (2) to $\mathbb{C}_p$, which gives its unique immediate maximally complete extension*

$$\mathbb{L}_p \coloneqq W(\overline{\mathbb{F}}_p)((p^{\mathbb{Q}})).$$

By applying Proposition 2.2 to $\mathbb{L}_p$, one knows that $\mathbb{L}_p$ is complete and algebraically closed. Moreover, one can show that $\mathbb{L}_p$ is much larger than $\mathbb{C}_p$:

Lemma 2.12 ([Poo93, Corollary 9]). *The field $\mathbb{L}_p$ has transcendence degree $2^{\aleph_0}$ over $\mathbb{C}_p$.*

2.2. **Basic properties of $\mathbb{L}_p$.** Compared to the unsatisfactoriness mentioned in Remark 2.3 (1), Kedlaya proved[3][4] the algebraic closeness of $\mathbb{L}_p$ by using a transfinite Newton algorithm as following:

For a non-constant polynomial $P(T) = \sum_{i=0}^{n} a_{n-i}T^i \in \mathbb{L}_p[T]$, denote by $\mathscr{N}\!ewt(P)$ the Newton polygon of P, i.e. the lower boundary of the convex hull of the set of points $(i, v_p(a_i))$ for $i = 0, 1, \cdots, n$. We write $s^P_{\max}$ for the slope of the segment of $\mathscr{N}\!ewt(P)$ with the largest slope and $m^P_{\max}$ the left endpoint of this segment. Besides that, call

$$\mathrm{Res}_P(T) \coloneqq \sum_{k=0}^{n-m^P_{\max}} C_{v_p(a_m)+s^P_{\max}(n-m^P_{\max}-k)}(a_{n-k})T^k$$

the residue polynomial of P, where for any $s \in \mathbb{Q}$, the map $C_s\colon \mathbb{L}_p \longrightarrow \overline{\mathbb{F}}_p$ is given by $\sum_{q\in\mathbb{Q}}[\zeta_q]p^q \longmapsto \zeta_s$.

We extracted Kedlaya's proof into the following pseudocode:

Algorithm 1 transfinite Newton algorithm for $\mathbb{L}_p$

INPUT: A non-constant polynomial $P(T) \in \mathbb{L}_p[T]$
OUTPUT: A root of $P(T)$ in $\mathbb{L}_p$

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  r ← 0, Φ(T) ← P(T)                ▷ We denote the coefficient of T^i in Φ as b_{n−i}.
  while Φ(0) ≠ 0 do                                ▷ This loop runs transfinitely.
      c ← any root of Res_Φ(T) in F̄_p
      r ← r + [c] · p^{s^Φ_max}
      Φ(T) ← Φ(T + [c] · p^{s^Φ_max})
  end while
  return r
```

We refer to [WY21] for a full explanation of this algorithm.

Let $r = \sum_{\omega}[\zeta_\omega]p^{k_\omega} \in \mathbb{L}_p$, with ordinal ω runs through the well-ordered set $\mathrm{supp}(r)$, be a root of $P(T)$ given by the above algorithm. For the convenience of later discussion, we call $r_\omega = \sum_{r<\omega}[\zeta_\omega]p^{k_\omega}$ the ω-th approximation of r, $P_\omega = P(T + r_\omega)$ the ω-th approximation polynomial and $\mathrm{Res}_{P_\omega}(T)$ the ω-th residue polynomial.

Example 2.13 ([WY21; WY23]). *Let $\zeta_{2(p-1)} \in W(\mathbb{F}_{p^2})$ be a $2(p-1)$-th primitive root of unity.*

(1) There exist a p-th root of unity, whose canonical expansion in $\mathbb{L}_p$ is given by

$$\zeta_p = \sum_{k=0}^{p-1} \frac{\zeta^k_{2(p-1)}}{k!} p^{\frac{k}{p-1}} + \sum_{k=p}^{\infty} [c_k] p^{\frac{k}{p-1}},$$

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[3]His proof is motivated by the work of Lampert (cf. [Lam86]).

[4]Actually Kedlaya's proof can be adapted to any Mal'cev-Neumann field (equal-characteristic or p-adic) with divisible value group and algebraically closed residue field.

where $c_k \in \mathbb{F}_{p^2}$ *for all* $k \geq p$.

(2) *For* $n \geq 2$*, there exists a* p^n*-th root of unity, whose (non-canonical) expansion in* $\mathbb{L}_p$ *is partially given by*

$$\begin{aligned}\zeta_{p^n} =& \sum_{k=0}^{p-1} \frac{(-1)^{nk}}{k!} \zeta_{2(p-1)}^k p^{\frac{k}{p^{n-1}(p-1)}} + \sum_{k=0}^{p-1} \frac{(-1)^{n(k+1)}}{k!} \zeta_{2(p-1)}^{k+1} p^{\frac{k+p}{p^{n-1}(p-1)}} \left(\sum_{l=n}^{\infty} p^{-1/p^l} \right) \\ &- \sum_{k=1}^{p-1} \frac{(-1)^{n(k+1)}}{k!} \left(\sum_{l=1}^{k} \frac{1}{l} \right) \zeta_{2(p-1)}^{k+1} p^{\frac{k+p}{p^{n-1}(p-1)}} \\ &+ \frac{1}{2} \zeta_{2(p-1)}^2 p^{\frac{2}{p^{n-2}(p-1)}} \left(\sum_{l=n}^{\infty} p^{-1/p^l} \right)^2 + \frac{(-1)^n}{2} \zeta_{2(p-1)}^3 p^{\frac{2}{p^{n-2}(p-1)} - \frac{p-2}{p^n(p-1)}} \\ &+ \text{ terms with higher valuation} \cdots .\end{aligned}$$

3. Field of hyper-algebraic elements in $\mathbb{L}_p$

3.1. Hyper-algebraic elements.

Definition 3.1. *We call an element* $f = \sum_{q \in \mathbb{Q}} [r_q] p^q \in \mathbb{L}_p$ ***hyper-algebraic****, if it satisfies:*

(1) *there exists a positive integer* N *such that* $\mathrm{supp}(f) \subseteq \frac{1}{N}\mathbb{Z}[1/p]$*;*

(2) *there exists a positive integer* k *such that* $r_q \in \mathbb{F}_{p^k}$ *for all* $q \in \mathrm{supp}(f)$.

Denote by $\mathbb{L}_p^{\mathrm{ha}}$ *the set of all hyper-algebraic elements in* $\mathbb{L}_p$.

By [Poo93, Corollary 8], we know that

Proposition 3.2. *The set* $\mathbb{L}_p^{\mathrm{ha}}$ *forms an algebraically closed field. As a consequence, all* p*-adic algebraic numbers are hyper-algebraic, i.e.* $\overline{\mathbb{Q}}_p \subseteq \mathbb{L}_p^{\mathrm{ha}}$.

We clarify the relations among the fields $\mathbb{L}_p^{\mathrm{ha}}$, $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$:

Theorem 3.3.

(1) *The fields* $\mathbb{L}_p^{\mathrm{ha}}$ *and* $\mathbb{C}_p$ *do not contain each other. In particular,* $\mathbb{L}_p^{\mathrm{ha}}$ *contains* $\overline{\mathbb{Q}}_p$ *as a proper subfield.*

(2) *The field* $\mathbb{L}_p^{\mathrm{ha}}$ *is not complete, and its completion is a proper subfield of* $\mathbb{L}_p$.

Proof. Consider the following element of $\mathbb{L}_p^{\mathrm{ha}}$:

$$\alpha = \sum_{k=1}^{\infty} p^{\frac{\lfloor \sqrt{2} \cdot p^k \rfloor}{p^k}}.$$

If $\alpha \in \mathbb{C}_p$, then there exists a p-adic algebraic number $\beta \in \overline{\mathbb{Q}}_p$ that $v_p(\alpha - \beta) > 2$. This shows that the canonical expansion of β in $\mathbb{L}_p^{\mathrm{ha}}$ has the form

$$\beta = \sum_{k=1}^{\infty} p^{\frac{\lfloor \sqrt{2} \cdot p^k \rfloor}{p^k}} + \text{ terms with exponent greater than 2.}$$

Thus, $\mathrm{supp}(\beta)$ has accumulation value $\sqrt{2}$. However, this is impossible: Lampert showed in [Lam86, Theorem 2] that the set

$$\mathcal{A} := \{\alpha \in \mathbb{L}_p | \{\text{accumulation value of } \mathrm{supp}(\alpha)\} \subset \mathbb{Q}\}$$

is an algebraically closed field. Since the support of every p-adic rational number lies in $\mathbb{Z} \subset \mathbb{Q}$, $\overline{\mathbb{Q}}_p$ is a subfield of $\mathcal{A}$. On the other hand, β does not belong to $\mathcal{A}$. This contradiction shows that $\mathbb{L}_p^{\mathrm{ha}}$ is not contained in $\mathbb{C}_p$. In particular, $\mathbb{L}_p^{\mathrm{ha}}$ contains $\overline{\mathbb{Q}}_p$ as a proper subfield.

To show that $\mathbb{L}_p^{\mathrm{ha}}$ is not complete and does not contain $\mathbb{C}_p$, we can consider the sequence $\left(\sum_{k=1}^{n} p^{k-1/k}\right)_{n\geq 1}$ in $\overline{\mathbb{Q}}_p \subseteq \mathbb{L}_p^{\mathrm{ha}}$, which clearly converges in $\mathbb{C}_p$ but has non-hyper-algebraic limit $\sum_{k=1}^{\infty} p^{k-1/k}$ in $\mathbb{L}_p$: the p-power-free part of the denominators of elements of its support is unbounded.

To prove $\mathbb{L}_p^{\mathrm{ha}}$ is not dense in $\mathbb{L}_p$, we consider the element $\gamma = \sum_{k=1}^{\infty} p^{-\frac{1}{k\cdot p^k}}$ in $\mathbb{L}_p$. If it lies in the completion of $\mathbb{L}_p^{\mathrm{ha}}$, then there exists an element $\delta \in \mathbb{L}_p^{\mathrm{ha}}$ that $v_p(\gamma - \delta) > 1$. This leads to a contradiction if we consider the canonical expansion of δ in $\mathbb{L}_p^{\mathrm{ha}}$

$$\delta = \sum_{k=1}^{\infty} p^{-\frac{1}{k\cdot p^k}} + \text{ terms with exponent greater than 1.}$$

The denominators of elements of $\mathrm{supp}(\delta)$ are unbounded, suggesting that δ is not hyper-algebraic.

□

3.2. **Hyper-tame index and hyper-inertia index.**

Definition 3.4. *Let $\theta = \sum_{q\in\mathbb{Q}} [r_q]p^q \in \mathbb{L}_p^{\mathrm{ha}}$ be a hyper-algebraic element in $\mathbb{L}_p$.*

(1) *Denote by $\mathfrak{T}_\theta$ the minimal positive integer e such that* $\mathrm{supp}(\theta) \subseteq \frac{1}{e}\mathbb{Z}[1/p]$. *We call it the **hyper-tame index** of θ.*
(2) *Denote by $\mathfrak{F}_\theta$ the minimal positive integer f such that $r_q \in \mathbb{F}_{p^f}$ for all $q \in \mathrm{supp}(\theta)$. We call it the **hyper-inertia index** of θ.*

*We call them the **hyper-algebraic invariants** of θ.*

The following lemmas collect several basic properties of the hyper-tame and hyper-inertia indices:

Lemma 3.5. *Let $\alpha = \sum_{q\in\mathbb{Q}} [r_q]p^q$ be a hyper-algebraic element in $\mathbb{L}_p$. Then one has*

(1) *the hyper-algebraic invariants $\mathfrak{T}_\alpha$ and $\mathfrak{F}_\alpha$ of α are coprime to p;*
(2) *If the set of coefficients $\{r_q\}_{q\in\mathbb{Q}}$ is contained in a finite field $\mathbb{F}_{p^s}$, then s is a multiplier of $\mathfrak{F}_\alpha$;*
(3) *If the support $\mathrm{supp}(\alpha)$ is contained in the set $\frac{1}{N}\mathbb{Z}[1/p]$ for some positive integer N, then N is a multiplier of $\mathfrak{T}_\alpha$;*

Proof.

(1) For any positive integer N, the sets $\frac{1}{pN}\mathbb{Z}[1/p]$ and $\frac{1}{N}\mathbb{Z}[1/p]$ are identical.
(2) One has

$$\{r_q\}_{q\in\mathbb{Q}} \subseteq \mathbb{F}_{p^{\mathfrak{F}_\alpha}} \cap \mathbb{F}_{p^s} = \mathbb{F}_{p^{\gcd(\mathfrak{F}_\alpha, s)}}.$$

The result follows from the minimality of $\mathfrak{F}_\alpha$.

(3) By the first assertion, we may assume that N is coprime to p. Suppose the contrary that $N = d \cdot \mathfrak{T}_\alpha + r$ with $d \in \mathbb{Z}_{\geq 1}$ and $r \in \{1, \cdots, \mathfrak{T}_\alpha - 1\}$. Take $q \in \mathrm{supp}(\alpha)$. Then the inclusion $q \in \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}[1/p] \cap \frac{1}{N}\mathbb{Z}[1/p]$ allows us to write

$$q = \frac{a_1 \cdot p^{v_1}}{\mathfrak{T}_\alpha} = \frac{a_2 \cdot p^{v_2}}{N},$$

where $a_1, a_2, v_1, v_2 \in \mathbb{Z}$ with a_1, a_2 coprime to p. By comparing the p-adic valuation, we get $v_1 = v_2$. Since

$$a_2 \cdot p^{v_2} = (d \cdot \mathfrak{T}_\alpha + r) \cdot q = d \cdot a_1 \cdot p^{v_1} + r \cdot q = d \cdot a_1 \cdot p^{v_2} + r \cdot q,$$

we obtain that $q = \frac{a_2 - d\cdot a_1}{r} \cdot p^{v_2} \in \frac{1}{r}\mathbb{Z}[1/p]$, which contradicts the minimality of $\mathfrak{T}_\alpha$.

□

Lemma 3.6. *Let $\alpha, \beta \in \mathbb{L}_p^{\mathrm{ha}}$ be two hyper-algebraic elements in $\mathbb{L}_p$. Then one has*

(1) $\mathfrak{T}_{\alpha+\beta} \mid \mathrm{lcm}(\mathfrak{T}_\alpha, \mathfrak{T}_\beta)$, $\mathfrak{F}_{\alpha+\beta} \mid \mathrm{lcm}(\mathfrak{F}_\alpha, \mathfrak{F}_\beta)$.
(2) $\mathfrak{T}_{\alpha\cdot\beta} \mid \mathrm{lcm}(\mathfrak{T}_\alpha, \mathfrak{T}_\beta)$, $\mathfrak{F}_{\alpha\cdot\beta} \mid \mathrm{lcm}(\mathfrak{F}_\alpha, \mathfrak{F}_\beta)$. *In particular if* α *is algebraic over* $\mathbb{Q}_p$ *and* $\mathbb{Q}_p(\alpha)$ *is unramified over* $\mathbb{Q}_p$, *then* $\mathfrak{T}_{\alpha\cdot\beta} \mid \mathfrak{T}_\beta$ *and* $\mathfrak{F}_{\alpha\cdot\beta} \mid \mathrm{lcm}(\mathfrak{f}_\alpha, \mathfrak{F}_\beta)$.
(3) $\mathfrak{T}_{1/\alpha} = \mathfrak{T}_\alpha$, $\mathfrak{F}_{1/\alpha} = \mathfrak{F}_\alpha$ *for* $\alpha \neq 0$.

Proof. The first and the second assertions follow from the definition of addition and multiplication on $\mathbb{L}_p$. In particular if $\mathbb{Q}_p(\alpha)$ is unramified over $\mathbb{Q}_p$, then $\mathbb{Q}_p(\alpha) = \mathrm{Frac}\, W(\mathbb{F}_{p^{\mathfrak{f}_\alpha}})$. As a result, every element in $\mathbb{Q}_p(\alpha)$ has the form $\sum_{k \gg -\infty} [\zeta_k] p^k$, where $\zeta_k \in \mathbb{F}_{p^{\mathfrak{f}_\alpha}}$ for all k. This shows that $\mathfrak{T}_\alpha = 1$ and $\mathfrak{F}_\alpha = \mathfrak{f}_\alpha$.

For the third assertion, the result is trivial when $|\operatorname{supp}(\alpha)| = 1$. Now we suppose $|\operatorname{supp}(\alpha)| \geq 2$ and write $\alpha = [\zeta] p^{v_p(\alpha)} - A$ for some $\zeta \in \overline{\mathbb{F}}_p$ with $v_p(A) > v_p(\alpha)$. Then $\zeta \in \mathbb{F}_{p^{\mathfrak{F}_\alpha}}$, $\mathfrak{T}_A \mid \mathfrak{T}_\alpha$ and $\mathfrak{F}_A \mid \mathfrak{F}_\alpha$. The result follows from the expansion

$$\alpha^{-1} = [\zeta^{-1}] p^{-v_p(\alpha)} \sum_{k=0}^{\infty} \Big([\zeta^{-1}] p^{-v_p(\alpha)} \cdot A \Big)^k,$$

where $v_p\big([\zeta^{-1}] p^{-v_p(\alpha)} \cdot A\big) > 0$, $\mathfrak{T}_{[\zeta^{-1}] p^{-v_p(\alpha)} \cdot A} \mid \mathfrak{T}_\alpha$ and $\mathfrak{F}_{[\zeta^{-1}] p^{-v_p(\alpha)} \cdot A} \mid \mathfrak{F}_\alpha$. □

Corollary 3.7. *For any positive integer* $e, f \geq 1$, *the set*

$$\mathbb{L}_p^{\mathrm{ha}}(e, f) \coloneqq \{\alpha \in \mathbb{L}_p^{\mathrm{ha}} \colon \mathfrak{F}_\alpha \mid f,\ \mathfrak{T}_\alpha \mid e\}$$

is a subfield of $\mathbb{L}_p^{\mathrm{ha}}$. *In particular, for any* $\alpha \in \mathbb{L}_p^{\mathrm{ha}}$, *we have* $\mathbb{Q}_p(\alpha) \subset \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$.

4. p-adic algebraic numbers in $\mathbb{L}_p^{\mathrm{ha}}$

The objective of this section is to investigate the hyper-algebraic invariants of p-adic algebraic numbers.

4.1. **Hyper-algebraic invariants of general p-adic algebraic numbers.** As observed in [Poo93, Corollary 8], there are two special types of automorphisms in $\mathrm{Aut}_{\mathbb{Q}_p}(\mathbb{L}_p)$:

(1) for any $g \in \mathcal{G}_{\mathbb{F}_p} \coloneqq \mathrm{Gal}\big(\overline{\mathbb{F}}_p/\mathbb{F}_p\big)$, it can be viewed as an element of $\mathrm{Aut}_{\mathbb{Q}_p}(\mathbb{L}_p)$ by the formula

$$g \cdot \sum_{q \in \mathbb{Q}} [r_q] p^q = \sum_{q \in \mathbb{Q}} [g(r_q)] p^q, \text{ for any } \sum_{q \in \mathbb{Q}} [r_q] p^q \in \mathbb{L}_p.$$

(2) For any group homomorphism $\xi \colon \mathbb{Q}/\mathbb{Z} \longrightarrow \overline{\mathbb{F}}_p^\times$, the following formula

$$\lambda_\xi \colon \sum_{q \in \mathbb{Q}} [r_q] p^q \longmapsto \sum_{q \in \mathbb{Q}} [\xi(q) r_q] p^q, \text{ for any } \sum_{q \in \mathbb{Q}} [r_q] p^q \in \mathbb{L}_p$$

also gives an element of $\mathrm{Aut}_{\mathbb{Q}_p}(\mathbb{L}_p)$.

Using these two types of automorphisms, we give a common upper bound of the hyper-algebraic invariants:

Theorem 4.1. *For every p-adic algebraic number* α, *one has* $\mathfrak{T}_\alpha, \mathfrak{F}_\alpha \leq [\mathbb{Q}_p(\alpha) \colon \mathbb{Q}_p]$.

Proof of Theorem 4.1 for hyper-inertia index. The action of $\mathcal{G}_{\mathbb{F}_p}$ on $\mathbb{L}_p$ sends p-adic algebraic numbers to their conjugates under the action of $\mathcal{G}_{\mathbb{Q}_p}$. As a result, one has

$$\Big|\{g(\alpha) \colon g \in \mathcal{G}_{\mathbb{F}_p}\}\Big| \leq [\mathbb{Q}_p(\alpha) \colon \mathbb{Q}_p] \tag{4.1}$$

for any $\alpha \in \overline{\mathbb{Q}}_p$.

Suppose there exists $\alpha \in \overline{\mathbb{Q}}_p$ that $\mathfrak{F}_\alpha > [\mathbb{Q}_p(\alpha) \colon \mathbb{Q}_p]$. Thus, there exists a rational number $q_0 \in \operatorname{supp}(\alpha)$ such that the minimal finite field containing $C_{q_0}(\alpha)$ is $\mathbb{F}_{p^r}$ with $r > [\mathbb{Q}_p(\alpha) \colon \mathbb{Q}_p]$. Consider the set

$$\{g(C_{q_0}(\alpha)) \colon g \in \mathcal{G}_{\mathbb{F}_p}\} = \Big\{ C_{q_0}(\alpha), C_{q_0}(\alpha)^p, \cdots, C_{q_0}(\alpha)^{p^n}, \cdots \Big\}.$$

The cardinality of this set is the minimal positive integer d that $C_{q_0}(\alpha) = C_{q_0}(\alpha)^{p^d}$, which is the same as the minimal positive integer d that $C_{q_0}(\alpha) \in \mathbb{F}_{p^d}$. This shows that $r = d$, i.e. $\left|\{g(C_{q_0}(\alpha)) \colon g \in \mathcal{G}_{\mathbb{F}_p}\}\right| = r$. Since the following map is surjective

$$\{g(\alpha) \colon g \in \mathcal{G}_{\mathbb{F}_p}\} \longrightarrow \{g(C_{q_0}(\alpha)) \colon g \in \mathcal{G}_{\mathbb{F}_p}\},\ g(\alpha) \longmapsto C_{q_0}(g(\alpha)) = g(C_{q_0}(\alpha)),$$

we know that $\left|\{g(\alpha) \colon g \in \mathcal{G}_{\mathbb{F}_p}\}\right| \geq r$, which contradicts to (4.1). □

We prove Theorem 4.1 for the hyper-tame index in the rest of this subsection. Denote by Set (resp. Ab) the category of sets (resp. abelian groups).

Definition 4.2. *Let M be a subset of $\mathbb{Q}/\mathbb{Z}$. A map $f \colon M \longrightarrow \overline{\mathbb{F}}_p^\times$ is called admissible, if it can be extended to a (non necessarily unique) group homomorphism $\widetilde{f} \in \mathrm{Hom}_{\mathrm{Ab}}\left(\mathbb{Q}/\mathbb{Z}, \overline{\mathbb{F}}_p^\times\right)$.*

For any $\alpha \in \mathbb{L}_p$, we denote by $\mathrm{Hom}_{\mathrm{Set}}^{\mathrm{adm}}\left(\mathrm{supp}(\alpha)/\mathbb{Z}, \overline{\mathbb{F}}_p^\times\right)$ the set of all admissible maps from $\mathrm{supp}(\alpha)/\mathbb{Z}$ to $\overline{\mathbb{F}}_p^\times$. If $\alpha = \sum_{q\in\mathbb{Q}}[r_q]p^q \in \overline{\mathbb{Q}}_p$ and $f \in \mathrm{Hom}_{\mathrm{Set}}^{\mathrm{adm}}\left(\mathrm{supp}(\alpha)/\mathbb{Z}, \overline{\mathbb{F}}_p^\times\right)$, then we have

$$\sum_{q\in\mathbb{Q}}[f(q)r_q]p^q = \sum_{q\in\mathbb{Q}}[\widetilde{f}(q)r_q]p^q \in \overline{\mathbb{Q}}_p$$

for any extension $\widetilde{f} \in \mathrm{Hom}_{\mathrm{Ab}}\left(\mathbb{Q}/\mathbb{Z}, \overline{\mathbb{F}}_p^\times\right)$. Note that for any group homomorphism $\xi \colon \mathbb{Q}/\mathbb{Z} \longrightarrow \overline{\mathbb{F}}_p^\times$, λ_ξ maps p-adic algebraic numbers to their conjugates under the action of $\mathcal{G}_{\mathbb{Q}_p} := \mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)$. This gives us an injective map:

$$\Phi_\alpha \colon \mathrm{Hom}_{\mathrm{Set}}^{\mathrm{adm}}\left(\mathrm{supp}(\alpha)/\mathbb{Z}, \overline{\mathbb{F}}_p^\times\right) \longrightarrow \{g(\alpha) \colon g \in \mathcal{G}_{\mathbb{Q}_p}\},\ f \longmapsto \lambda_{\widetilde{f}}(\alpha).$$

Lemma 4.3. *Let A be a subset of $\mathbb{Q}$ and let $\langle A/\mathbb{Z}\rangle$ be the subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by $A/\mathbb{Z}$. Then*

(1) One has a bijection:

$$\mathrm{Hom}_{\mathrm{Ab}}\left(\langle A/\mathbb{Z}\rangle, \overline{\mathbb{F}}_p^\times\right) \longrightarrow \mathrm{Hom}_{\mathrm{Set}}^{\mathrm{adm}}\left(A/\mathbb{Z}, \overline{\mathbb{F}}_p^\times\right).$$

(2) If $\mathrm{Hom}_{\mathrm{Set}}^{\mathrm{adm}}\left(A/\mathbb{Z}, \overline{\mathbb{F}}_p^\times\right)$ is a finite set, then $A \subseteq \frac{1}{N}\mathbb{Z}[1/p]$, where $N = \left|\mathrm{Hom}_{\mathrm{Ab}}\left(\langle A/\mathbb{Z}\rangle, \overline{\mathbb{F}}_p^\times\right)\right|$.

Proof.

(1) By restricting the morphisms in $\mathrm{Hom}_{\mathrm{Ab}}\left(\langle A/\mathbb{Z}\rangle, \overline{\mathbb{F}}_p^\times\right)$ to $A/\mathbb{Z}$, we obtain an injection

$$\iota \colon \mathrm{Hom}_{\mathrm{Ab}}\left(\langle A/\mathbb{Z}\rangle, \overline{\mathbb{F}}_p^\times\right) \longrightarrow \mathrm{Hom}_{\mathrm{Set}}\left(A/\mathbb{Z}, \overline{\mathbb{F}}_p^\times\right).$$

We are left to show that the image of this map is exactly $\mathrm{Hom}_{\mathrm{Set}}^{\mathrm{adm}}\left(A/\mathbb{Z}, \overline{\mathbb{F}}_p^\times\right)$.

For any $f \in \mathrm{Hom}_{\mathrm{Set}}^{\mathrm{adm}}\left(A/\mathbb{Z}, \overline{\mathbb{F}}_p^\times\right)$, any extension $\widetilde{f} \in \mathrm{Hom}_{\mathrm{Ab}}\left(\mathbb{Q}/\mathbb{Z}, \overline{\mathbb{F}}_p^\times\right)$ of f has image f by the injection ι. This implies that $\mathrm{Hom}_{\mathrm{Set}}^{\mathrm{adm}}\left(A/\mathbb{Z}, \overline{\mathbb{F}}_p^\times\right)$ is contained in the image of ι.

For any $h = \iota(a) \in \mathrm{Hom}_{\mathrm{Set}}\left(A/\mathbb{Z}, \overline{\mathbb{F}}_p^\times\right)$ with some $a \in \mathrm{Hom}_{\mathrm{Ab}}\left(\langle A/\mathbb{Z}\rangle, \overline{\mathbb{F}}_p^\times\right)$, a extends uniquely to a group homomorphism $\widetilde{a} \in \mathrm{Hom}_{\mathrm{Ab}}\left(\mathbb{Q}/\mathbb{Z}, \overline{\mathbb{F}}_p^\times\right)$ since $\overline{\mathbb{F}}_p^\times$ is an injective object in Ab. Since $\widetilde{a}\mid_{A/\mathbb{Z}} = a\mid_{A/\mathbb{Z}} = \iota(a) = h$, we know that h is admissible.

(2) The following proof is given by Lahtonen (cf. [Lah24]). Let $N = |\operatorname{Hom}_{\mathrm{Ab}}\Big(\langle A/\mathbb{Z}\rangle, \overline{\mathbb{F}}_p^{\times}\Big)|$. Suppose there exists a rational number $q \in \mathbb{Q}$ that $q + \mathbb{Z} \in \langle A/\mathbb{Z}\rangle$ and $q \notin \frac{1}{N}\mathbb{Z}[1/p]$. We write $q = \frac{u}{p^r \cdot v}$, where $u, v \in \mathbb{Z}_{\geq 1}$, $r \in \mathbb{Z}_{\geq 0}$ with $\gcd(u, v) = \gcd(p, v) = 1$. Since $q \notin \frac{1}{N}\mathbb{Z}[1/p]$, one knows that v does not divide N.

Notice that the element $z' := \frac{u}{v} + \mathbb{Z}$ has order v in $\langle A/\mathbb{Z}\rangle \subseteq \mathbb{Q}/\mathbb{Z}$. Fix a v-th primitive root ζ_v of unity in $\overline{\mathbb{F}}_p^{\times}$, then the map $z' \longmapsto \zeta_v$ induces a morphism d in $\operatorname{Hom}_{\mathrm{Ab}}\Big(\langle z'\rangle, \overline{\mathbb{F}}_p^{\times}\Big)$ with order v. Since $\overline{\mathbb{F}}_p^{\times}$ is injective in Ab, d extends to a morphism $\widetilde{d} \in \operatorname{Hom}_{\mathrm{Ab}}\Big(\langle A/\mathbb{Z}\rangle, \overline{\mathbb{F}}_p^{\times}\Big)$. The order of $\widetilde{d}$ in $\operatorname{Hom}_{\mathrm{Ab}}\Big(\langle A/\mathbb{Z}\rangle, \overline{\mathbb{F}}_p^{\times}\Big)$, which divides N by Lagrange's theorem, is a multiplier of v. This contradicts to the assertion that v does not divide N. Thus, $\langle A/\mathbb{Z}\rangle \subset \frac{1}{N}\mathbb{Z}[1/p]/\mathbb{Z}$, which allows us to conclude the proof.

□

Proof of Theorem 4.1 for hyper-tame index. We can set A in Lemma 4.3 (2) to be $\operatorname{supp}(\alpha)$, and we obtain $\operatorname{supp}(\alpha) \subseteq \frac{1}{N}\mathbb{Z}[1/p]$, where

$$N = \Big|\operatorname{Hom}_{\mathrm{Ab}}\Big(\langle \operatorname{supp}(\alpha)/\mathbb{Z}\rangle, \overline{\mathbb{F}}_p^{\times}\Big)\Big|.$$

By Lemma 4.3 (1), we have $N = \Big|\operatorname{Hom}^{\mathrm{adm}}_{\mathrm{Set}}\Big(\operatorname{supp}(\alpha)/\mathbb{Z}, \overline{\mathbb{F}}_p^{\times}\Big)\Big|$. Thus, $\mathfrak{T}_\alpha \leq N \leq [\mathbb{Q}_p(\alpha) \colon \mathbb{Q}_p]$, as promised. □

Remark 4.4. *One should not expect that $\mathfrak{T}_\alpha$ divides $[\mathbb{Q}_p(\alpha) \colon \mathbb{Q}_p]$ for general p-adic algebraic number α. To see this, consider $\alpha = p^{1/p} \cdot \zeta_p$, which has hyper-tame degree $\mathfrak{T}_\alpha = p - 1$ while $[\mathbb{Q}_p(\alpha) \colon \mathbb{Q}_p] = p$.*

4.2. **Hyper-algebraic invariants of abelian extensions.** Let ζ_{p^n} be the p^n-th root of unity in Example 2.13. It is easy to see that

	$\alpha = \zeta_p$	$\alpha = \zeta_{p^n}$ $(n \geq 2)$
$\mathfrak{F}_\alpha$	2	≥ 2
$\mathfrak{T}_\alpha$	$p - 1$	$\geq p - 1$

.

The following proposition gives a precise form of the above observations:

Proposition 4.5. *For any integer $n \geq 1$ and any p^n-th primitive root of unity ζ_{p^n}, we have $\mathfrak{T}_{\zeta_{p^n}} = p - 1$ and*

$$\mathfrak{F}_{\zeta_{p^n}} \begin{cases} = \quad 2, & \text{if } n = 1, 2; \\ \text{divides } 2 \cdot p^{n-2}, & \text{if } n \geq 3. \end{cases}$$

The key to prove this proposition is the following lemma:

Lemma 4.6. *Let $\alpha \in \mathbb{L}_p^{\mathrm{ha}}$ with $v_p(\alpha) = 0$. Then there exists a p-th root β of α in $\mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, p \cdot \mathfrak{F}_\alpha)$. In particular, if $C_{\frac{1}{p-1}}(\beta) = 0$, then β belongs to $\mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$.*

Proof. We apply the transfinite Newton algorithm on the equation $T^p - \alpha = 0$ to get a root β. Set $\beta = \sum_\omega [c_\omega] \cdot p^{k_\omega}$, where the ordinal ω runs through the well-ordered set $\operatorname{supp}(\beta)$. Recall that for any ordinal ω, let $\beta_\omega = \sum_{\rho<\omega} [c_\rho] \cdot p^{k_\rho}$ and

$$\Phi_\omega(T) = (T + \beta_\omega)^p - \alpha = T^p + \sum_{k=1}^{p-1} \binom{p}{k} \beta_\omega^k \cdot T^{p-k} + \beta_\omega^p - \alpha.$$

The first step is easy: since $\beta_0 = 0$ and $\Phi_0(T) = T^p - \alpha$, the Newton polygon $\mathcal{N}ewt(\Phi_0)$ consists of a single horizontal segment with residue polynomial given by

$$\mathrm{Res}_{\Phi_0}(T) = T^p - C_0(\alpha) \in \mathbb{F}_{p^{\mathfrak{F}_\alpha}}[T],$$

which splits in $\mathbb{F}_{p^{\mathfrak{F}_\alpha}}$. This shows that $\beta_1 \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$ and $v_p(\beta_1) = 0$.

For any $\omega \geq 1$, since $v_p(\beta_\omega) = v_p(\beta_1) = 0$, we know that $v_p(\binom{p}{k}\beta_\omega^k) = 1$ for all $k = 1, 2, \cdots, p-1$. This implies that $\mathcal{N}ewt(\Phi_\omega)$ is determined by the point $(p, v_p(\beta_\omega^p - \alpha))$ for every $\omega \geq 1$.

Since $k_\omega \in \mathbb{Q}$ increases monotonically with respect to the ordinal ω, we set ω_0 to be the minimal ordinal ρ that satisfies $k_\rho \geq \frac{1}{p-1}$.

(1) Suppose $\omega < \omega_0$ and $\beta_\rho \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$ for every $\rho \leq \omega$. Then $\mathcal{N}ewt(\Phi_\omega)$ consists of a single segment with slope $k_\omega = s_{\max}^{\Phi_\omega} = \frac{1}{p} v_p(\beta_\omega^p - \alpha) < \frac{1}{p-1}$.

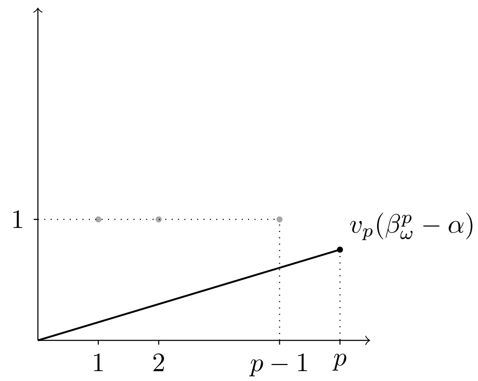

FIGURE 4.1. $\mathcal{N}ewt(\Phi_\omega)$, $1 \leq \omega < \omega_0$

Since $\beta_\omega^p - \alpha \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$ by Corollary 3.7, we know that

$$v_p(\beta_\omega^p - \alpha) \in \mathrm{supp}(\beta_\omega^p - \alpha) \subseteq \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}[1/p].$$

This implies that $k_\omega = \frac{1}{p} v_p(\beta_\omega^p - \alpha)$ also belongs to $\frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}[1/p]$. The residue polynomial of $\Phi_\omega(T)$ is given by

$$\mathrm{Res}_{\Phi_\omega}(T) = T^p + C_{v_p(\beta_\omega^p - \alpha)}(\beta_\omega^p - \alpha),$$

where $C_{v_p(\beta_\omega^p - \alpha)}(\beta_\omega^p - \alpha) \in \mathbb{F}_{p^{\mathfrak{F}_\alpha}}$. Thus, any root of this residue polynomial lies in $\mathbb{F}_{p^{\mathfrak{F}_\alpha}}$. This shows that $\beta_{\omega+1} \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$. Since the case of limit ordinals is self-indicating, we can show by transfinite induction that $\beta_\omega \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$ for all $\omega \leq \omega_0$.

(2) Now we deal with $\omega = \omega_0 + 1$.

(a) If $k_{\omega_0} = s_{\max}^{\Phi_{\omega_0}} = \frac{1}{p-1}$, then $\mathcal{N}ewt(\Phi_{\omega_0})$ consists of a single segment with slope equals to

$$k_{\omega_0} = \frac{1}{p-1} = \frac{1}{p} v_p(\beta_{\omega_0}^p - \alpha) \in \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}[1/p].$$

Since this segment contains the point $(p-1, 1)$, one knows that

$$\mathrm{Res}_{\Phi_{\omega_0}}(T) = T^p + C_0(\beta_{\omega_0})^{p-1}T + C_{v_p(\beta_{\omega_0}^p - \alpha)}(\beta_{\omega_0}^p - \alpha) \in \mathbb{F}_{p^{\mathfrak{T}_\alpha}}[T],$$

whose root lies in $\mathbb{F}_{p^{p \cdot \mathfrak{F}_\alpha}}$. In this case, one has $\beta_{\omega_0+1} \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, p \cdot \mathfrak{F}_\alpha)$.

(b) If $k_{\omega_0} = s_{\max}^{\Phi_{\omega_0}} > \frac{1}{p-1}$, then $\mathscr{N}ewt(\Phi_{\omega_0})$ consists of two segments, where the vertexes of the segment with maximal slope is given by $(p-1, 1)$ and $(p, v_p(\beta_{\omega_0}^p - \alpha))$. Thus,

$$k_{\omega_0} = \frac{v_p(\beta_{\omega_0}^p - \alpha) - 1}{p - (p-1)} \in \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}[1/p]$$

and one has

$$\mathrm{Res}_{\Phi_{\omega_0}}(T) = C_0(\beta_{\omega_0})^{p-1}T + C_{v_p(\beta_{\omega_0}^p - \alpha)}(\beta_{\omega_0}^p - \alpha),$$

whose root lies in $\mathbb{F}_{p^{\mathfrak{F}_\alpha}}$. In this case, one has $\beta_{\omega_0+1} \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_\alpha)$.

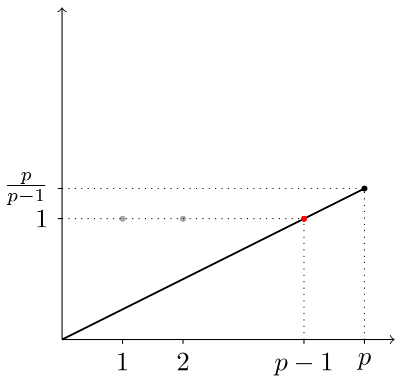

Figure 4.2. $\mathscr{N}ewt(\Phi_{\omega_0})$, if $k_{\omega_0} = \frac{1}{p-1}$

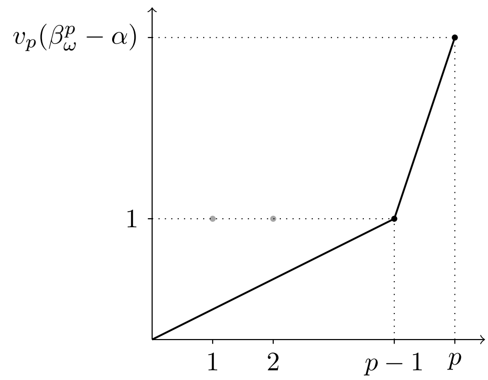

Figure 4.3. $\mathscr{N}ewt(\Phi_{\omega_0})$, if $k_{\omega_0} > \frac{1}{p-1}$

(3) For the case of $\omega > \omega_0$, we have $k_\omega > \frac{1}{p-1}$. With the same calculation as above, one can prove by transfinite induction that for any ordinal $\omega \geq \omega_0 + 1$, $\beta_\omega \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_\alpha, \mathfrak{F}_{\beta_{\omega_0+1}})$.

The result follows. □

Additionally, we need the following auxiliary lemma:

Lemma 4.7. *For any p^2-th primitive root of unity ζ_{p^2}, there exists another p^2-th primitive root of unity ζ'_{p^2} and a p-th root of unity ξ_c (not necessarily primitive) that $\zeta_{p^2} = \zeta'_{p^2} \cdot \xi_c$ and $C_{\frac{1}{p-1}}(\zeta'_{p^2}) = 0$.*

Proof. Fix a $2(p-1)$-th primitive root of unity $\tilde{\zeta}_{2(p-1)}$. Let

$$\mathcal{W} := \left\{\tilde{\zeta}_{2(p-1)}^{2k+1} : k \in \mathbb{N}_{<p-1}\right\}.$$

By choosing $\zeta_{2(p-1)}$ in the expansion of the p^2-th primitive root of unity given by Example 2.13 (see also [WY21, Theorem 3.3]) in $\mathcal{W}$, we get $p-1$ different p^2-th primitive roots of unity $r_0, r_1, \cdots, r_{p-2}$, satisfying $[C_{\frac{1}{p(p-1)}}(r_k)] = \tilde{\zeta}_{2(p-1)}^{2k+1}$ and $[C_{\frac{1}{p-1}}(r_k)] = 0$ for every $k \in \mathbb{N}_{<p-1}$.

Similarly, for every $c \in \{0\} \cup \mathcal{W}$, there exists a p-th root of unity (not necessarily primitive) ξ_c that $v_p\Big(\xi_c - 1 - c \cdot p^{\frac{1}{p-1}}\Big) > \frac{1}{p-1}$. Thus, for any $k \in \mathbb{N}_{<p-1}$ and $c \in \{0\} \cup \mathcal{W}$, $r_k \cdot \xi_c$ is a p^2-th primitive root of unity, satisfying $[C_{\frac{1}{p(p-1)}}(r_k \cdot \xi_c)] = \tilde{\zeta}_{2(p-1)}^{2k+1}$ and $[C_{\frac{1}{p-1}}(r_k \cdot \xi_c)] = c$. This enumerates all $p(p-1)$ p^2-th primitive roots of unity. The result follows. □

Proof of Proposition 4.5. The case of $n = 1$ follows immediately from [WY21, Proposition 3.4].

Let ζ_{p^2} be any p^2-th primitive root of unity. By Lemma 4.7, there exists another p^2-th primitive root of unity ζ'_{p^2} and a p-th root of unity ξ_c (not necessarily primitive) that $\zeta_{p^2}^p = \zeta'_{p^2} \cdot \xi_c$ and $C_{\frac{1}{p-1}}(\zeta'_{p^2}) = 0$. By applying Lemma 4.6, we have

$$\zeta'_{p^2} \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{T}_{(\zeta'_{p^2})^p}, \mathfrak{F}_{(\zeta'_{p^2})^p}) = \mathbb{L}_p^{\mathrm{ha}}(p-1, 2).$$

Since $\xi_c \in \mathbb{L}_p^{\mathrm{ha}}(p-1,2)$, we know that $\zeta_{p^2} \in \mathbb{L}_p^{\mathrm{ha}}(p-1,2)$. On the other hand, by [WY21, Theorem 3.3], one has $\mathfrak{T}_{\zeta_{p^2}} \geq p-1$ and $\mathfrak{F}_{\zeta_{p^2}} \geq 2$. This implies that $\mathfrak{T}_{\zeta_{p^2}} = p-1$ and $\mathfrak{F}_{\zeta_{p^2}} = 2$.

When $n \geq 3$, we can set $\alpha = (\zeta_{p^n})^p$ in Lemma 4.6 inductively to get the result. One should notice that when $n \geq 3$, we no longer know if the analog of Lemma 4.7 holds for ζ_{p^n}. Thus, the hyper-inertia index is multiplied by p when n increases by 1. □

Corollary 4.8. *For any positive integer $m = r \cdot p^{v_p(m)}$ with $\gcd(r,p) = 1$ and any m-th primitive root of unity ζ_m, one has*

(1) If $v_p(m) = 0$, then $\mathfrak{T}_{\zeta_m} = 1$ and $\mathfrak{F}_{\zeta_m} = \mathrm{ord}_r\, p$.
(2) If $v_p(m) \geq 1$, then $\mathfrak{T}_{\zeta_m} \mid p-1$ and

$$\mathfrak{F}_{\zeta_m} \mid \begin{cases} \mathrm{lcm}(2, \mathrm{ord}_r\, p), & \text{if } v_p(m) = 1, 2; \\ \mathrm{lcm}\big(2 \cdot p^{v_p(m)-1}, \mathrm{ord}_r\, p\big), & \text{if } v_p(m) \geq 3. \end{cases}$$

Proof. It suffices to note that any r-th root of unity lies in $W(\mathbb{F}_{p^{\mathrm{ord}_r p}})$. □

With the power of the local Kronecker-Weber theorem, we can generalize this result to those p-adic algebraic numbers that generate abelian extensions over $\mathbb{Q}_p$:

Theorem 4.9. *Let $\alpha \in \overline{\mathbb{Q}}_p$ be a p-adic algebraic number with $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ an abelian extension of degree n. Denote by $\mathbf{f}_{\mathbb{Q}_p(\alpha)}$ the local conductor of $\mathbb{Q}_p(\alpha)$ over $\mathbb{Q}_p$. Then*

(1) If $\mathbf{f}_{\mathbb{Q}_p(\alpha)} = 0$, then $\mathfrak{T}_\alpha = 1$ and $\mathfrak{F}_\alpha = n$.
(2) If $\mathbf{f}_{\mathbb{Q}_p(\alpha)} \geq 1$, then $\mathfrak{T}_\alpha \mid p-1$ and

$$\mathfrak{F}_\alpha \mid \begin{cases} \mathrm{lcm}(2, n), & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} = 1, 2; \\ \mathrm{lcm}\Big(2 \cdot p^{\mathbf{f}_{\mathbb{Q}_p(\alpha)}-1}, n\Big), & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} \geq 3. \end{cases}.$$

To prove this theorem, the following effective form of the local Kronecker-Weber theorem is needed:

Lemma 4.10. *Let $K/\mathbb{Q}_p$ be an abelian extension of degree n with conductor $\mathbf{f}_K$ and let $m = (p^n - 1)p^{\mathbf{f}_K}$. Then $K \subseteq \mathbb{Q}_p(\zeta_m)$.*

Proof. By [Gui18, Lemma 4.11] and its proof, there exists $s \geq 1$ that

$$\langle p^s \rangle \times U_{\mathbb{Q}_p}^{(\mathbf{f}_K)} \subseteq \mathcal{N}_{K/\mathbb{Q}_p} K^\times.$$

It follows that $K \subseteq \mathbb{Q}_p\Big(\zeta_{(p^s-1)p^{\mathbf{f}_K}}\Big)$ by the proof of [Gui18, Theorem 13.27]. On the other hand, we have $K \subseteq \mathbb{Q}_p\Big(\zeta_{(p^n-1)p^{v_p(n)+2}}\Big)$ by [KS22, Theorem 3.1]. Since

$$\mathbb{Q}_p\Big(\zeta_{(p^s-1)p^{\mathbf{f}_K}}\Big) \cap \mathbb{Q}_p\Big(\zeta_{(p^n-1)p^{v_p(n)+2}}\Big) \subseteq \mathbb{Q}_p(\zeta_m),$$

we have $K \subseteq \mathbb{Q}_p(\zeta_m)$. □

Proof of Theorem 4.9. Let $m = (p^n - 1)p^{\mathbf{f}_{\mathbb{Q}_p(\alpha)}}$. By Lemma 4.10, we know that $\alpha \in \mathbb{Q}_p(\zeta_m)$.

Note $\operatorname{ord}_{p^n-1} p = n$. By Corollary 4.8, we know that

$$\mathfrak{T}_{\zeta_m} = \begin{cases} 1, & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} = 0; \\ p-1, & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} \geq 1, \end{cases}$$

and

$$\mathfrak{F}_{\zeta_m} \begin{cases} = \quad n, & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} = 0; \\ = \quad \operatorname{lcm}(2, n), & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} = 1, 2; \\ \text{divides } \operatorname{lcm}\Big(2 \cdot p^{\mathbf{f}_{\mathbb{Q}_p(\alpha)} - 1}, n\Big), & \text{if } \mathbf{f}_{\mathbb{Q}_p(\alpha)} \geq 3. \end{cases}$$

Since $\alpha \in \mathbb{Q}_p(\zeta_m) \subseteq \mathbb{L}_p^{\text{ha}}(\mathfrak{T}_{\zeta_m}, \mathfrak{F}_{\zeta_m})$, the result follows. □

4.3. **Criterion for tamely ramified extensions.**

Theorem 4.11. *Let $\alpha \in \mathbb{L}_p^{\text{ha}}$ be a hyper-algebraic element in $\mathbb{L}_p$. Then $\mathbb{Q}_p(\alpha)$ is tamely ramified over $\mathbb{Q}_p$ if and only if* $\operatorname{supp}(\alpha) \subseteq \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}$. *In this situation, we have $\mathfrak{T}_\alpha = \mathfrak{e}_\alpha$, $\mathfrak{f}_\alpha \mid \mathfrak{F}_\alpha$ and $\mathfrak{F}_\alpha \mid c$, where $c := \operatorname{ord}_{\operatorname{lcm}(\mathfrak{e}_\alpha, p^{\mathfrak{f}_\alpha}-1)} p$ and $\mathfrak{f}_\alpha$ (resp. $\mathfrak{e}_\alpha$) is the inertia degree (resp. the ramification index) of the extension $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$.*

The proof of this theorem relies on the following lemma:

Lemma 4.12. *Let $\alpha \in \overline{\mathbb{Q}}_p$ be a p-adic algebraic number with $\mathbb{Q}_p(\alpha)$ tamely ramified over $\mathbb{Q}_p$. Then there exists a $\mathfrak{e}_\alpha$-th root $\zeta_e \in \overline{\mathbb{F}}_p$ of unity that*

$$\mathbb{Q}_p(\alpha) = \mathbb{Q}_{p^{\mathfrak{f}_\alpha}}\Big(p^{1/\mathfrak{e}_\alpha} \cdot [\zeta_e]\Big),$$

where $\mathbb{Q}_{p^{\mathfrak{f}_\alpha}} := W(\mathbb{F}_{p^{\mathfrak{f}_\alpha}})\left[\frac{1}{p}\right]$ is the maximal unramified extension of $\mathbb{Q}_p$ in $\mathbb{Q}_p(\alpha)$.

Proof. Let $\mathcal{O}_K$ be the ring of integer of $K := \mathbb{Q}_p(\alpha)$ with a uniformizer π_K. Suppose $\pi_K^{\mathfrak{e}_\alpha} = p \cdot u$, where u is a unit in $\mathcal{O}_K^\times$.

Note that the polynomial $T^{\mathfrak{e}_\alpha} - \overline{u} \in \mathbb{F}_{p^{\mathfrak{f}_\alpha}}[T]$ has simple roots by the condition $\gcd(\mathfrak{e}_\alpha, p) = 1$. Hensel lemma implies that there is a $\mathfrak{e}_\alpha$-th root v of u in $\mathcal{O}_K^\times$. If we set $\pi_K' := \pi_K \cdot v^{-1}$, then this element is also a uniformizer of K. Since π_K' is a $\mathfrak{e}_\alpha$-th root of p, we have $\pi_K' = p^{1/\mathfrak{e}_\alpha} \cdot [\zeta_e]$ for some $\mathfrak{e}_\alpha$-th root ζ_e of unity in $\overline{\mathbb{F}}_p$. □

Proof of Theorem 4.11. If $\operatorname{supp}(\alpha) \subseteq \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}$, we can write $\alpha = \sum_{k \gg -\infty}^{+\infty} [r_k] \cdot p^{\frac{k}{\mathfrak{T}_\alpha}}$, where $r_k \in \mathbb{F}_{p^{\mathfrak{F}_\alpha}}$ for all k. Thus, α lies in $\mathbb{Q}_{p^{\mathfrak{F}_\alpha}}\Big(p^{\frac{1}{\mathfrak{T}_\alpha}}\Big)$, where $\mathbb{Q}_{p^{\mathfrak{F}_\alpha}} := W(\mathbb{F}_{p^{\mathfrak{F}_\alpha}})\left[\frac{1}{p}\right]$ is the unique unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_{p^{\mathfrak{F}_\alpha}}$. Since $\mathfrak{T}_\alpha$ is coprime to p (cf. Lemma 3.5), the field $\mathbb{Q}_{p^{\mathfrak{F}_\alpha}}\Big(p^{\frac{1}{\mathfrak{T}_\alpha}}\Big)$ is tamely ramified over $\mathbb{Q}_p$, implying that $\mathbb{Q}_p(\alpha)$ is also tamely ramified over $\mathbb{Q}_p$.

Conversely, if $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ is tamely ramified, then we have

$$\mathbb{Q}_p(\alpha) = \mathbb{Q}_{p^{\mathfrak{f}_\alpha}}\Big(p^{1/\mathfrak{e}_\alpha} \cdot [\zeta_e]\Big)$$

for some $\mathfrak{e}_\alpha$-th root $\zeta_e \in \overline{\mathbb{F}}_p$ of unity by Lemma 4.12. Let

$$\alpha = \sum_{k=0}^{\mathfrak{e}_\alpha - 1} c_k \cdot \Big(p^{1/\mathfrak{e}_\alpha} \cdot [\zeta_e]\Big)^k$$

with $c_k \in \mathbb{Q}_{p^{\mathfrak{f}_\alpha}}$ for $k = 0, \cdots, \mathfrak{e}_\alpha - 1$. If we set $c_k = \sum_{i>-\infty} \left[c_i^{(k)}\right] p^i \in \mathbb{Q}_{p^{\mathfrak{f}_\alpha}}$ with $c_i^{(k)} \in \mathbb{F}_{p^{\mathfrak{f}_\alpha}}$, then

$$\alpha = \sum_{k=0}^{\mathfrak{e}_\alpha - 1} \sum_{i>-\infty} \left[c_i^{(k)} \cdot \zeta_e^k\right] p^{i+k/\mathfrak{e}_\alpha}. \tag{4.2}$$

This shows that $\operatorname{supp}(\alpha) \subseteq \frac{1}{\mathfrak{e}_\alpha}\mathbb{Z}$. Thus,

$$\operatorname{supp}(\alpha) \subseteq \frac{1}{\mathfrak{e}_\alpha}\mathbb{Z} \cap \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}[1/p] \subseteq \mathbb{Z}_{(p)} \cap \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}[1/p] = \frac{1}{\mathfrak{T}_\alpha}\mathbb{Z}.$$

To prove the second assertion, notice that the inclusion $\alpha \in \mathbb{Q}_{p^{\mathfrak{F}_\alpha}}\left(p^{\frac{1}{\mathfrak{T}_\alpha}}\right)$ implies $\mathfrak{e}_\alpha \mid \mathfrak{T}_\alpha$ and $\mathfrak{f}_\alpha \mid \mathfrak{F}_\alpha$. On the other hand, if any coefficient $c_i^{(k)} \cdot \zeta_e^k$ in (4.2) is non-zero, then it is a $\operatorname{lcm}(\mathfrak{e}_\alpha, p^{\mathfrak{f}_\alpha} - 1)$-th root of unity, i.e. $c_i^{(k)} \cdot \zeta_e^k \in \mathbb{F}_{p^c}$. As a result, one conclude by Lemma 3.5 that $\alpha \in \mathbb{L}_p^{\mathrm{ha}}(\mathfrak{e}_\alpha, c)$. □

Compared to Theorem 4.1, the constant c in Theorem 4.11 does provide a better bound for the hyper-inertia index in the tamely ramified case:

Lemma 4.13. *Let $c := \operatorname{ord}_{\operatorname{lcm}(\mathfrak{e}_\alpha, p^{\mathfrak{f}_\alpha}-1)} p$ be the constant in Theorem 4.11. Then c divides $\operatorname{lcm}(\phi(\mathfrak{e}_\alpha), \mathfrak{f}_\alpha)$, where ϕ is Euler's totient function.*

Proof. Let $e_0 := \frac{\mathfrak{e}_\alpha}{\gcd(\mathfrak{e}_\alpha, p^{\mathfrak{f}_\alpha}-1)}$, then $\operatorname{lcm}(\mathfrak{e}_\alpha, p^{\mathfrak{f}_\alpha} - 1) = e_0 \cdot (p^{\mathfrak{f}_\alpha} - 1)$, with e_0 a factor of $\mathfrak{e}_\alpha$ that coprime to $p^{\mathfrak{f}_\alpha} - 1$ and p. Chinese remainder theorem implies that

$$c = \operatorname{lcm}(\operatorname{ord}_{e_0} p, \operatorname{ord}_{p^{\mathfrak{f}_\alpha}-1} p) = \operatorname{lcm}(\operatorname{ord}_{e_0} p, \mathfrak{f}_\alpha).$$

Since e_0 is a factor of $\mathfrak{e}_\alpha$, we have $\operatorname{ord}_{e_0} p$ divides $\operatorname{ord}_{\mathfrak{e}_\alpha} p$. The result follows from Euler's theorem. □

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