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ON THE p-ADIC TRANSCENDENCE OF $\sum_{k=1}^{\infty} p^{-1/p^k}$

by

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Abstract. — Let p be a prime number. In this article, we prove that the p-adic Hahn series $\sum_{k=1}^{\infty} p^{-1/p^k}$, which is the mixed-characteristic analogue of Abhyankar's solution $\sum_{k=1}^{\infty} t^{-1/p^k}$ to the Artin-Schreier equation $X^p - X - t^{-1} = 0$ over $\mathbf{F}_p((t))$, is a p-adic complex number, but not a p-adic algebraic number. Based on this result, we formulate a conjecture about the possible order type of the support of an algebraic p-adic Hahn series and prove that it is implied by a tentative observation of Kedlaya.

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1. Introduction

In analogy with classical transcendental number theory, which concerns algebraicity over $\mathbf{Q}$, the study of function fields and p-adic fields has likewise developed into an important and well-established branch of transcendental number theory.

When k is an algebraically closed field of characteristic 0, the Puiseux-Newton theorem states that the algebraic closure of $k((t))$ is precisely the field of Puiseux series $k\left(\left(t^{1/\infty}\right)\right) = \bigcup_{n=1}^{\infty} k\left(\left(t^{1/n}\right)\right)$. In contrast, the structure of the algebraic closure becomes significantly more intricate when the base field k has positive characteristic. For example, Chevalley showed in [Che51, p. 64] that the Artin-Schreier equation

(a) $$X^p - X - t^{-1} = 0$$

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over $\mathbf{F}_p(\!(t)\!)$ (and even $\overline{\mathbf{F}}_p(\!(t)\!)$) has no solution in the field of Puiseux series $\overline{\mathbf{F}}_p\big(\!\big(t^{1/\infty}\big)\!\big)$. On the other hand, Abhyankar observed in [Abh56] that the formal series

$$\mathfrak{a} := \sum_{k=1}^{\infty} t^{-1/p^k}, \tag{b}$$

which lives in the field of Hahn series $\overline{\mathbf{F}}_p\big(\!\big(t^{\mathbf{Q}}\big)\!\big)$ (cf. Example 2.5), is a root of the above equation. This example demonstrates that the framework of Hahn series offers an appropriate setting for investigating transcendental number theory over local fields of equal characteristic $p > 0$. In 2001, Kedlaya gives a sufficient and necessary condition (cf. [Ked01b, Theorem 8, Corollary 9][(1)]) for a Hahn series in $\overline{\mathbf{F}}_p\big(\!\big(t^{\mathbf{Q}}\big)\!\big)$ (resp. $\mathbf{F}_p\big(\!\big(t^{\mathbf{Q}}\big)\!\big)$) to be algebraic over $\overline{\mathbf{F}}_p(\!(t)\!)$ (resp. $\mathbf{F}_p(\!(t)\!)$) with the language of twist-recurrent sequences.

A comparable phenomenon is also observed in the mixed-characteristic context. As proved by Lampert (cf. [Lam86]), Poonen (cf. [Poo93]) and Kedlaya (cf. [Ked01a]), for any prime number p, the field $\mathscr{O}_{\breve{\mathbf{Q}}_p}\big(\!\big(p^{\mathbf{Q}}\big)\!\big)$ of p-adic Hahn series (cf . Example 2.5) with residue field $\overline{\mathbf{F}}_p$ and value group $\mathbf{Q}$ is the spherical completion of $\overline{\mathbf{Q}}_p$, which is algebraically closed and complete. By passing to Witt vectors, Kedlaya's criterion generalizes to verify whether a p-adic Hahn series lies in the **completed** integral closure of $\mathscr{O}_{\breve{\mathbf{Q}}_p}$, i.e. $\mathscr{O}_{\mathbf{C}_p}$ (cf. Theorem 3.1). For example, we consider the p-adic analogue of Abhyankar's series (b):

$$\mathfrak{A} := \sum_{k=1}^{\infty} p^{-1/p^k} \in \mathscr{O}_{\breve{\mathbf{Q}}_p}\big(\!\big(p^{\mathbf{Q}}\big)\!\big).$$

Kedlaya's result allows us to prove

***Theorem A* (cf. Theorem 3.1).** — *The p-adic Hahn series $\mathfrak{A}$ lies in $\mathbf{C}_p$.*

Owing to the completeness nature of Witt vectors, Kedlaya's criterion is unable to ascertain whether a p-adic Hahn series is algebraic over $\mathbf{Q}_p$. There are several necessaries conditions for a p-adic Hahn series to be algebraic over $\mathbf{Q}_p$. For example, in [Lam86], [Ked01a] and [WY24], it is shown that if a p-adic Hahn series $\sum_{q\in\mathbf{Q}}[c_q]p^q$ is algebraic over $\mathbf{Q}_p$, then

1. There exists $r \in \mathbf{N}_{\geq 1}$ such that $c_q \in \mathbf{F}_{p^r}$ for all $q \in \mathbf{Q}$;
2. There exists $N \in \mathbf{N}_{\geq 1}$ such that the support $\{q \in \mathbf{Q} \mid c_q \neq 0\}$ is contained in $\frac{1}{N}\mathbf{Z}[p^{-1}]$;
3. The accumulation points of the support $\{q \in \mathbf{Q} \mid c_q \neq 0\}$ are rational numbers.

Note that the element $\mathfrak{A}$ satisfies all above necessary conditions, and it is even p-quasi-automatic in the sense of Kedlaya [Ked17]. In contrast to Abhyankar's observation regarding the algebracity of $\mathfrak{a}$, we establish the following result, which is counterintuitive:

***Theorem B* (cf. Theorem 4.4).** — *The p-adic Hahn series $\mathfrak{A}$ is transcendental over $\mathbf{Q}_p$.*

This finding highlights the fundamental distinction between transcendental number theory in the context of local fields with equal characteristic $p > 0$ and that pertaining to local fields of mixed characteristic $(0, p)$.

Due to the inherent difficulty in establishing a general criterion for the transcendence of p-adic Hahn series over $\mathbf{Q}_p$, more tractable problems are proposed. For example, Lampert asked in [Lam86] about the possible order type of the support of an $\mathbf{Q}_p$-algebraic p-adic Hahn series. Inspired by the transcendence of $\mathfrak{A}$, which has bounded but infinite support, we formutate the following conjecture:

(1) See also [Ked17, Remark 2.9] for the critical remark on [Ked01b, Theorem 8].

***Conjecture C* (cf. Conjecture 5.1).** — *If a p-adic algebraic number has bounded support, then its support is finite.*

We will prove in Proposition 5.2 that this conjecture is implied by a hypothetical observation by Kedlaya in [Ked01a]. It is anticipated that this conjecture will constitute a foundational advancement in the study of the $\mathbf{Q}_p$-transcendence properties of p-adic Hahn series.

2. Preliminaries on Hahn series

To make this article self-contained, we briefly recall some basic facts about Hahn series.

***Definition 2.1* ([Poo93, Section 3]).** — *Let R be a commutative ring and G be an ordered group.*

1. *For any $f \in \mathrm{Hom}_{\mathrm{Set}}(G, R)$, we define the* ***support*** *of f to be*
$$\mathrm{Supp}(f) = \{g \in G \colon f(g) \neq 0\}.$$
2. *Define the set of* ***Hahn series*** *over R with value group G to be*
$$R(\!(G)\!) := \{f \in \mathrm{Hom}_{\mathrm{Set}}(G, R) \colon \mathrm{Supp}(f) \text{ is well-ordered}\}.$$
By introducing a formal variable t, elements in $R(\!(G)\!)$ will also be written as $\sum_{g\in G} r_g t^g$, where $r_g \in R$ for all $g \in G$.

***Proposition 2.2* ([Poo93, Lemma 1, Corollary 2]).** — *Let R be a commutative ring and G be an ordered group.*

1. *With identity $1 \cdot t^0$ and addition as well as multiplication given by*
$$\sum_{g\in G} a_g t^g + \sum_{g\in G} b_g t^g := \sum_{g\in G} (a_g + b_g) t^g, \ \sum_{g\in G} a_g t^g \cdot \sum_{g\in G} b_g t^g := \sum_{g\in G} \left(\sum_{h\in G} a_h b_{g-h}\right) t^g$$
$R(\!(G)\!)$ forms a commutative ring.
2. *If R is a field, then so does $R(\!(G)\!)$. Moreover, with the map*
$$v \colon R(\!(G)\!) \longrightarrow G \cup \{\infty\}, \ f \longmapsto \begin{cases} \min \mathrm{Supp}(f), & \text{if } f \neq 0 \\ \infty, & \text{if } f = 0 \end{cases},$$
$R(\!(G)\!)$ becomes a valued field with value group G and residue field R.

Note that $\mathrm{char}\, R(\!(G)\!) = \mathrm{char}\, R$, we call $R(\!(G)\!)$ the **equal-characteristic field of Hahn series** over R with value group G, also denoted as $R\big(\!\big(t^G\big)\!\big)$ with respect to the formal variable t.

***Proposition 2.3* ([Poo93, Proposition 3, Corollary 3, Proposition 5])**

Let k be a perfect field of characteristic p and G be an ordered group containing $\mathbb{Z}$ as a subgroup. Besides that, let
$$\mathcal{N} := \left\{ \sum_{g\in G} r_g t^g \in W(k)\big(\!\big(t^G\big)\!\big) \colon \text{ for every } g \in G, \ \sum_{n\in\mathbb{Z}} r_{g+n} p^n = 0 \right\},$$
where $W(k)$ is the ring of Witt vectors of k. Then

1. *$\mathscr{N}$ is a maximal ideal of $W(k)(\!(t^G)\!)$, which makes $W(k)(\!(p^G)\!) := W(k)(\!(t^G)\!)/\mathscr{N}$ a field*(2)*, called the* ***p-adic field of Hahn series****.*
2. *Every element in $W(k)(\!(p^G)\!)$ can be uniquely written as*
$$\sum_{g\in G}[r_g]p^g,$$
where $r_g \in k$ for all $g \in G$ and $[\cdot]\colon k \longrightarrow W(k)$ is the Teichmüller lift. We call this the ***standard expansion*** *of the element.*
3. *For $f = \sum_{g\in G}[r_g]p^g$, define the* ***support*** *of f to be*
$$\mathrm{Supp}(f) = \{g \in G \colon r_g \neq 0\}.$$
Then the map
$$v\colon W(k)(\!(G)\!)/\mathscr{N} \longrightarrow G \cup \{\infty\},\ f \mapsto \begin{cases} \min \mathrm{Supp}(f), & \text{if } f \neq 0 \\ \infty, & \text{if } f = 0 \end{cases}$$
makes $W(k)(\!(G)\!)/\mathscr{N}$ a mixed-characteristic valued field with value group G and residue field k.

The most important property of the field of Hahn series is the following:

Theorem 2.4 **(cf. [Poo93, Theorem 1, Corollary 4, Corollary 6])**

Let F be an equal-characteristic (resp. mixed-characteristic) valued field with divisible value group G and algebraically closed residue field k. Then the equal-characteristic (resp. p-adic) field of Hahn series $k(\!(t^G)\!)$ (resp. $W(k)(\!(p^G)\!)$) is the unique (up to isomorphisms of valued field) minimal spherically complete extension of F. Moreover, it is algebraically closed and complete.

The following examples are the fields of Hahn series used in this article:

Example 2.5. — *Let $F = \overline{\mathbf{F}}_p(\!(t)\!)$ (resp. $\breve{\mathbf{Q}}_p = W(\overline{\mathbf{F}}_p)[p^{-1}]$), which has value group $\mathbf{Q}$ and residue field $\overline{\mathbf{F}}_p$. Then the field of equal-characteristic (resp. p-adic) Hahn series $\overline{\mathbf{F}}_p(\!(t^{\mathbf{Q}})\!)$ (resp. $\mathscr{O}_{\breve{\mathbf{Q}}_p}(\!(p^{\mathbf{Q}})\!) = W(\overline{\mathbf{F}}_p)(\!(p^{\mathbf{Q}})\!)$) is the spherical completion of F with residue field and value group unchanged, which is algebraically closed and complete.*

Remark 2.6. — *To simplify the statement, we will simply call $\mathscr{O}_{\breve{\mathbf{Q}}_p}(\!(p^{\mathbf{Q}})\!)$ the p-adic Hahn series without specifying the residue field and value group.*

We end this section by proving the following lemma, which will be used in the proof of Theorem 4.4:

Lemma 2.7. — *Every p-adic Hahn series can be uniquely written as $\sum_{q\in\mathbf{Q}\cap[0,1)} c_q \cdot p^q$, with $c_q \in \breve{\mathbf{Q}}_p$ for all q.*

Proof. — Notice that $\mathbf{Q} \cap [0, 1)$ is a set of representatives of $\mathbf{Q}/\mathbf{Z}$, every p-adic Hahn series $\sum_{q\in\mathbf{Q}}[a_q]p^q$ can be written as
$$\sum_{q\in\mathbf{Q}}[a_q]p^q = \sum_{q\in\mathbf{Q}\cap[0,1)}\sum_{n\in\mathbf{Z}}[a_{q+n}]p^{q+n} = \sum_{q\in\mathbf{Q}\cap[0,1)} p^q\left(\sum_{n\in\mathbf{Z}}[a_{q+n}]p^n\right),$$
where $\sum_{n\in\mathbf{Z}}[a_{q+n}]p^n \in \breve{\mathbf{Q}}_p$ for all q.

(2) Intuitively speaking, $W(k)(\!(p^G)\!)$ is obtained by replacing the formal variable t of elements in $W(k)(\!(t^G)\!)$ by the prime p.

For the uniqueness, for any p-adic Hahn series $\sum_{q\in\mathbf{Q}}[a_q]p^q$, write

$$\sum_{q\in\mathbf{Q}}[a_q]p^q = \sum_{q\in\mathbf{Q}\cap[0,1)} c_q\cdot p^q = \sum_{q\in\mathbf{Q}\cap[0,1)} d_q\cdot p^q$$

with $c_q, d_q \in \breve{\mathbf{Q}}_p$ for all $q\in\mathbf{Q}\cap[0,1)$. Then we have

$$0 = \sum_{q\in\mathbf{Q}\cap[0,1)} (c_q - d_q)p^q.$$

If we write $c_q - d_q = \sum_{n\in\mathbf{Z}}[s_{q,n}]p^n \in \breve{\mathbf{Q}}_p$ with $s_{q,n}\in\overline{\mathbf{F}}_p$ for all n and for all $q\in\mathbf{Q}\cap[0,1)$, then

$$0 = \sum_{q\in\mathbf{Q}\cap[0,1)} \sum_{n\in\mathbf{Z}}[s_{q,n}]p^{q+n}. \tag{c}$$

Since $q+n$ are all distinct for different pairs (q,n), (c) is the standard expansion of $0\in\mathscr{O}_{\breve{\mathbf{Q}}_p}(\!(p^{\mathbf{Q}})\!)$. Hence $s_{q,n}=0$ for all $q\in\mathbf{Q}\cap[0,1)$ and $n\in\mathbf{Z}$, i.e. $c_q = d_q$ for all $q\in\mathbf{Q}\cap[0,1)$. □

3. $\mathfrak{A}$ is a p-adic complex number

Although the summation $\sum_{k=1}^{\infty} p^{-1/p^k}$ is not interpreted as the p-adic limit (which does not exist) of the suspicious sequence $\left\{\sum_{k=1}^{n} p^{-1/p^k}\right\}_{n\geq 1}\subset\overline{\mathbf{Q}}_p$, we can still prove that it is a p-adic limit of a sequence in $\overline{\mathbf{Q}}_p$. This is a direct consequence of the following result of Kedlaya:

***Theorem 3.1* (cf. [Ked17, Theorem 13.4]).** — *The ring $\mathscr{O}_{\mathbf{C}_p}$ equals to the following sets:*

1. *the completion of the image of the p-quasi-automatic elements (cf. [Ked17, Definition 6.3, Definition 13.1]) of $W(\overline{\mathbf{F}}_p)(\!(t^{\mathbf{Q}})\!)^{\wedge}$ under the projection*
$$W(\overline{\mathbf{F}}_p)(\!(t^{\mathbf{Q}})\!)^{\wedge} \longrightarrow \mathscr{O}_{\breve{\mathbf{Q}}_p}(\!(p^{\mathbf{Q}})\!).$$
2. *the completion of the following set:*
$$\left\{\sum_{q\in\mathbf{Q}}[c_q]p^q\in\mathscr{O}_{\breve{\mathbf{Q}}_p}(\!(p^{\mathbf{Q}})\!)\,\middle|\,\sum_{q\in\mathbf{Q}} c_q\cdot t^q\in\widehat{\mathscr{O}}_L\right\},$$
where $\widehat{\mathscr{O}}_L$ is the completion of the integral closure of $\overline{\mathbf{F}}_p[\![t]\!]$ in $\overline{\mathbf{F}}_p(\!(t^{\mathbf{Q}})\!)$.

Remark 3.2. — *In the original statement of [Ked17, Theorem 13.4], the ring $\widehat{\mathscr{O}}_L$ (resp. $\mathscr{O}_{\mathbf{C}_p}$) is described as the "completed integral closure" of the field $\overline{\mathbf{F}}_p(\!(t)\!)$ (resp. $W(\overline{\mathbf{F}}_p)[p^{-1}]=\breve{\mathbf{Q}}_p$). By comparing [Ked17, Theorem 11.12] with [Ked17, Definition 6.3], we believe that the "completed integral closure" of a valued field F in a larger valued field E in [Ked17] means the completion of the integral closure of $\mathscr{O}_F$ in E, rather than the completion of the integral closure (which is also the algebraic closure) of F in E, which is the literal interpretation of "completed integral closure".*

As an application, we can prove that $\mathfrak{A}$ can be approximated by a Cauchy sequence in $\overline{\mathbf{Q}}_p$:

Proposition 3.3. — *The p-adic Hahn series $\mathfrak{A}=\sum_{k=1}^{\infty} p^{-1/p^k}$ lies in $\mathbf{C}_p$.*

Proof. — One can verify by direct calculation that the Hahn series $\sum_{k=1}^{\infty} t^{1-1/p^k}$ is a root of the polynomial

$$X^p - t^{p-1}X - t^{p-1}\in\overline{\mathbf{F}}_p[\![t]\!][X].$$

By Theorem 3.1, we conclude that $p\cdot\mathfrak{A}=\sum_{k=1}^{\infty} p^{1-1/p^k}$ lies in $\mathbf{C}_p$, and the result follows. □

Remark 3.4. — *The proof of Proposition 3.3 is not effective, i.e. it does not provide an explicit Cauchy sequence in $\overline{\mathbf{Q}}_p$ that converges to $\mathfrak{A}$. By Kedlaya's transfinite Newton algorithm (cf. [Ked01a, Theorem 1], [WY21, Section 2.2]), we mention two examples of p-adic algebraic numbers, whose p-adic distance to $\mathfrak{A}$ is less than 1:*

1. *There exists a root α of the Artin-Schreier polynomial $X^p - X - p^{-1} \in \mathbf{Q}_p[X]$, which is the p-adic analogue of* (a), *such that*
$$\alpha = \sum_{k=1}^{\infty} p^{-1/p^k} + p^{1/p - 1/p^2} + \text{ terms with higher valuation},$$
i.e. $v_p(\alpha - \mathfrak{A}) = \frac{1}{p} - \frac{1}{p^2}$.
2. *When $p \geq 3$, then [WY23] shows that for any integer $n \geq 2$, there exists a p^n-th root of unity ζ_{p^n} with expansion*
$$\zeta_{p^n} = \sum_{k=0}^{p-1} \frac{\zeta_{2(p-1)}^k}{[k!]} p^{\frac{k}{p^{n-1}(p-1)}} + \zeta_{2(p-1)} p^{\frac{1}{p^{n-2}(p-1)}} \sum_{k=n}^{\infty} p^{-1/p^k}$$
$$+ \zeta_{2(p-1)}^2 p^{\frac{1}{p^{n-2}(p-1)} + \frac{1}{p^n(p-1)}} + \text{ terms with higher valuation},$$
where $\zeta_{2(p-1)}$ is a fixed primitive $2(p-1)$-th root of unity. If we set
$$\beta_n := \left(\zeta_{2(p-1)}^2 \cdot p^{\frac{1}{p^{n-2}(p-1)}}\right)^{-1} \cdot \left(\zeta_{p^n} - \sum_{k=0}^{p-1} \frac{\zeta_{2(p-1)}^k}{[k!]} p^{\frac{k}{p^{n-1}(p-1)}}\right) + \sum_{k=1}^{n-1} p^{-1/p^k},$$
then
$$v_p(\beta_n - \mathfrak{A}) = \frac{1}{p^n(p-1)}.$$

4. $\mathbf{Q}_p$-transcendence of $\mathfrak{A}$

To show that $\mathfrak{A}$ is transcendental over $\mathbf{Q}_p$, we prove by contradiction by assuming that $\mathfrak{A}$ is the root of a polynomial f with $\mathbf{Z}_p$ coefficients. Our strategy is straightforward: we are going to show that there are some terms in the multinomial expansion of the leanding term of $f(\mathfrak{A})$ is impossible to be cancelled out by any other terms in the whole expansion. To rigiously carry out this idea, we need some preparations.

Definition 4.1. — *Let $\mathbb{I} := \bigoplus_{\mathbf{N}_{>0}} \mathbf{N}$.*

1. *Let $\lambda \colon \mathbb{I} \longrightarrow \mathbf{Q}$, $(a_k)_{k\geq 1} \longmapsto \sum_{k=1}^{\infty} -\frac{a_k}{p^k}$.*
2. *Let $\Sigma \colon \mathbb{I} \longrightarrow \mathbf{N}$, $(a_k)_{k\geq 1} \longmapsto \sum_{k=1}^{\infty} a_k$.*
3. *Let $\kappa \colon \mathbb{I} \longrightarrow \mathbf{N}$, $(a_k)_{k\geq 1} \longmapsto \max\{k \geq 1 \mid x_k > p-1\}$, where we set $\max \emptyset = 0$. This always makes sense since for every $\underline{a} = (a_k)_{k\geq 1} \in \mathbb{I}$, there are only finitely many k's with $a_k \neq 0$.*
4. *We call an element $\underline{a} \in \mathbb{I}$ **reduced**, if $\kappa(\underline{a}) = 0$, i.e. $0 \leq a_k \leq p-1$ for all $k \geq 1$.*
5. *For any $\underline{a}, \underline{b} \in \mathbb{I}$, write $\underline{a} \sim \underline{b}$ if $\lambda(\underline{a}) - \lambda(\underline{b}) \in \mathbf{Z}$. It is easy to see that $\sim$ is an equivalence relation.*

With above notations, we can write the multinomial expansion of $\mathfrak{A}^i$, $i = 0, \cdots, n+1$ as

$$\mathfrak{A}^i = \sum_{\substack{\underline{k} \in \mathbb{I} \\ \Sigma(\underline{k}) = i}} s_i \binom{i}{\underline{k}} p^{\lambda(\underline{k})}. \tag{d}$$

Intuitively, the "cancellation" could only happen between terms $p^{\lambda(\underline{a})}$ and $p^{\lambda(\underline{b})}$ with $\underline{a} \sim \underline{b}$. Hence we need to understand the equivalence classes of $\sim$ in $\mathbb{I}$. The following lemma shows that each equivalence class contains a unique reduced element, which plays the role of the "canonical representative" of the equivalence class.

Lemma 4.2. —

1. *For every $\underline{a} \in \mathbb{I}$, there exists a unique reduced element $\underline{a}_{\text{red}} \in \mathbb{I}$ such that $\underline{a} \sim \underline{a}_{\text{red}}$.*
2. *One has $\Sigma(\underline{a}_{\text{red}}) \leq \Sigma(\underline{a})$. The equality holds if and only if $\underline{a}$ is reduced, i.e. $\underline{a} = \underline{a}_{\text{red}}$.*

Proof. — We prove by induction on $\kappa(\underline{a})$ that there exists a reduced element $\underline{a}_{\text{red}} \in \mathbb{I}$ such that $\underline{a} \sim \underline{a}_{\text{red}}$ and $\Sigma(\underline{a}_{\text{red}}) \leq \Sigma(\underline{a})$.

If $\kappa(\underline{a}) = 0$, then we can take $\underline{a}_{\text{red}} = \underline{a}$. Suppose that $\kappa(\underline{a}) = n + 1$ for certain $n \in \mathbf{N}$ and the claim holds for all elements in $\mathbb{I}$ with κ-value not greater than n.

Since $a_{n+1} > p - 1$, we can write $a_{n+1} = r + p \cdot d$, with $r \in \{0, \cdots, p-1\}$ and $d \in \mathbf{N}_{>0}$. Let $\underline{a}' := (a'_k)_{k \geq 1} \in \mathbb{I}$ be defined as follows:

$$a'_k := \begin{cases} a_k, & \text{if } k \neq n, n+1; \\ r, & \text{if } k = n+1; \\ a_n + d, & \text{if } n \geq 1 \text{ and } k = n. \end{cases}$$

Then one can check by direct calculation that

$$\lambda(\underline{a}) = -\frac{r + p \cdot d}{p^{n+1}} + \sum_{k \neq n+1} -\frac{a_k}{p^k} = -\frac{r}{p^{n+1}} - \frac{a_n + d}{p^n} + \sum_{k \neq n, n+1} -\frac{a_k}{p^k} = \lambda(\underline{a}')$$

when $n \geq 1$, and

$$\lambda(\underline{a}) = -\frac{r + p \cdot d}{p} + \sum_{k \geq 2} -\frac{a_k}{p^k} = \lambda(\underline{a}') - d$$

when $n = 0$. In both cases, we have $\underline{a} \sim \underline{a}'$ and $\Sigma(\underline{a}') \leq n$. By the induction hypothesis, there exists a reduced element $\underline{a}'_{\text{red}} \in \mathbb{I}$ such that $\underline{a}' \sim \underline{a}'_{\text{red}}$ and $\Sigma(\underline{a}'_{\text{red}}) \leq \Sigma(\underline{a}')$. Hence we have $\underline{a} \sim \underline{a}'_{\text{red}}$ and $\Sigma(\underline{a}'_{\text{red}}) \leq \Sigma(\underline{a})$. This finishes the proof of the existence.

Now take two reduced elements $\underline{x}$ and $\underline{y}$ with $\underline{x} \sim \underline{y}$. Since

$$-1 = -\sum_{k=1}^{\infty} \frac{p-1}{p^k} < \lambda(\underline{x}), \lambda(\underline{y}) \leq 0,$$

the condition $\underline{x} \sim \underline{y}$ implies that $\lambda(\underline{x}) = \lambda(\underline{y})$. By viewing $\lambda(\underline{x})$ and $\lambda(\underline{y})$ as floating point numbers in base p, a digit-by-digit comparison shows that they are equal. This finishes the proof of the uniqueness.

For the last statement, suppose that $\Sigma(\underline{a}_{\text{red}}) = \Sigma(\underline{a})$. If $\Sigma(\underline{a}) = 0$, then $\underline{a} = \underline{a}_{\text{red}} = (0, 0, \cdots)$. Now we assume $\Sigma(\underline{a}) > 0$ and $\underline{a}$ is not reduced. Then the construction in the first part of the proof shows that there exists $\underline{a}' \in \mathbb{I}$ with $\underline{a}' \sim \underline{a}$ and $\Sigma(\underline{a}') < \Sigma(\underline{a})$. As a result, we obtain $\Sigma(\underline{a}_{\text{red}}) \leq \Sigma(\underline{a}') < \Sigma(\underline{a})$, which contradicts the assumption. □

Corollary 4.3. — *For every $\underline{a}, \underline{b} \in \mathbb{I}$, $\underline{a} \sim \underline{b}$ if and only if $\underline{a}_{\text{red}} = \underline{b}_{\text{red}}$.*

Now we are ready to prove the main result of this article:

Theorem 4.4. — *The p-adic Hahn series $\mathfrak{A} = \sum_{k=1}^{\infty} p^{-1/p^k}$ is transcendental over $\mathbf{Q}_p$.*

Proof. — Suppose the contrary that there exists $n \in \mathbf{N}$ and $s_0, \cdots, s_{n+1} \in \mathbf{Z}_p$ such that

$$s_0 + s_1\mathfrak{A} + \cdots + s_n\mathfrak{A}^n + s_{n+1}\mathfrak{A}^{n+1} = 0,$$

with s_0 and s_{n+1} nonzero. The multinomial theorem implies that

$$0 = \sum_{i=0}^{n+1} s_i \mathfrak{A}^i = \sum_{i=0}^{n+1} \sum_{\substack{\underline{k}\in\mathbb{I} \\ \Sigma(\underline{k})=i}} s_i \binom{i}{\underline{k}} p^{\lambda(\underline{k})} = \sum_{\substack{\underline{k}\in\mathbb{I} \\ \Sigma(\underline{k})\le n+1}} s_{\Sigma(\underline{k})} \binom{\Sigma(\underline{k})}{\underline{k}} p^{\lambda(\underline{k})}.$$

If we group the terms on the right hand side according to the equivalence classes of $\sim$, we obtain

$$\text{(e)} \qquad \begin{aligned} 0 &= \sum_{\substack{\underline{k}_{\mathrm{red}}\in\mathbb{I} \\ \underline{k}_{\mathrm{red}} \text{ reduced}}} \left(\sum_{\substack{\underline{k}\in\mathbb{I} \\ \Sigma(\underline{k})\le n+1 \\ \underline{k}\sim\underline{k}_{\mathrm{red}}}} s_{\Sigma(\underline{k})} \binom{\Sigma(\underline{k})}{\underline{k}} p^{\lambda(\underline{k})} \right) \\ &= \sum_{\substack{\underline{k}_{\mathrm{red}}\in\mathbb{I} \\ \underline{k}_{\mathrm{red}} \text{ reduced}}} p^{\lambda(\underline{k}_{\mathrm{red}})+\delta_0} \left(\sum_{\substack{\underline{k}\in\mathbb{I} \\ \Sigma(\underline{k})\le n+1 \\ \underline{k}\sim\underline{k}_{\mathrm{red}}}} s_{\Sigma(\underline{k})} \binom{\Sigma(\underline{k})}{\underline{k}} p^{\lambda(\underline{k})-\lambda(\underline{k}_{\mathrm{red}})-\delta_0} \right), \end{aligned}$$

where

$$\delta_0 = \begin{cases} 0, & \text{if } \underline{k}_{\mathrm{red}} = (0,0,\cdots); \\ 1, & \text{otherwise.} \end{cases}$$

Observe that for different reduced elements $\underline{k}_{\mathrm{red}} \in \mathbb{I}$, the values $\lambda(\underline{k}_{\mathrm{red}}) + \delta_0 \in [0,1)$ are all distinct, and if $\underline{k} \sim \underline{k}_{\mathrm{red}}$, then $\lambda(\underline{k}) - \lambda(\underline{k}_{\mathrm{red}}) - \delta_0 \in \mathbf{N}$. Hence Lemma 2.7 allows us to conclude from (e) that

$$\text{(f)} \qquad \sum_{\substack{\underline{k}\in\mathbb{I} \\ \Sigma(\underline{k})\le n+1 \\ \underline{k}\sim\underline{k}_{\mathrm{red}}}} s_{\Sigma(\underline{k})} \binom{\Sigma(\underline{k})}{\underline{k}} p^{\lambda(\underline{k})-\lambda(\underline{k}_{\mathrm{red}})} = 0$$

for every reduced element $\underline{k}_{\mathrm{red}} \in \mathbb{I}$. If we take the reduced element

$$\underline{k}^* := (\overbrace{1,1,\cdots,1}^{n+1 \text{ times}}, 0, \cdots,) \in \mathbb{I},$$

then for every $\underline{k} \in \mathbb{I}$ with $\Sigma(\underline{k}) \le n+1$ and $\underline{k} \sim \underline{k}^*$, Lemma 4.2 implies that $\Sigma(\underline{k}) = \Sigma(\underline{k}^*) = n+1$ and consequently $\underline{k} = \underline{k}^*$. Hence (f) can be specialized to the case of $\underline{k}_{\mathrm{red}} = \underline{k}^*$ as

$$s_{n+1} \binom{n+1}{\underline{k}^*} = s_{n+1} \cdot (n+1)! = 0,$$

which contradicts the assumption that s_{n+1} is nonzero. □

Remark 4.5. — *One does not need to worry about the well-definedness of any of the infinite sums above, since for every exponent with value of the form $\lambda(\underline{k})$, there are only finitely many terms contributing to it (cf. [Poo93, Lemma 1]).*

Remark 4.6. — *After some necessary generalization of λ, Σ and κ, the same idea can potentially be applied to show the $\mathbf{Q}_p$-transcendence of more general p-adic Hahn series.*

5. Order type of p-adic algebraic numbers

In [Lam86], Lampert asked about the possible order type of the support of the expansion of p-adic algebraic numbers as p-adic Hahn series. Kedlaya showed in [Ked01a, Section 4] that the order type can not exceed ω^ω. He also made the following remark at the end of same section:

> In fact, it is entirely possible that the answer to Lampert's question is that only finite orders, ω and ω^ω can occur.

We simply call it **Kedlaya's prediction**. Note that this prediction indicates that there does not exist any p-adic algebraic number with bounded support of order type ω: if $\alpha = \sum_{k=0}^{\infty}[c_k]p^{r_k} \in \mathscr{O}_{\check{\mathbf{Q}}_p}(\!(p^{\mathbf{Q}})\!)$ is a p-adic algebraic number with $\operatorname{diam}(\operatorname{Supp}(\alpha)) < N$ for certain positive integer N, then

$$\frac{\alpha}{1-p^N} = \sum_{k=0}^{\infty}[c_k]p^{r_k} + \sum_{k=0}^{\infty}[c_k]p^{r_k+N} + \sum_{k=0}^{\infty}[c_k]p^{r_k+2N} + \cdots$$

is a p-adic algebraic number with order type ω^2, which contradicts Kedlaya's prediction.

Since currently no p-adic algebraic number with bounded support of infinite order type is known, and even simple p-adic Hahn series like $\mathfrak{A}$ is transcendental over $\mathbf{Q}_p$, we raise the following conjecture about the order type of p-adic algebraic numbers:

Conjecture 5.1. — *If a p-adic algebraic number has bounded support, then its support is finite.*

In fact, this conjecture is also implicated by Kedlaya's prediction:

Proposition 5.2. — *Kedlaya's prediction implies Conjecture 5.1.*

Proof. — Suppose x is a p-adic algebraic number with bounded support of order type α. Let N be a positive integer such that $\operatorname{diam}(\operatorname{Supp}(x)) < N$. Similar to the argument above, we can show that

$$\frac{x}{1-p^N} = \sum_{t=0}^{\infty} x \cdot p^{Nt}$$

is a p-adic algebraic number with order type $\alpha \cdot \omega$. If we write α as its Cantor normal form (cf. [Sie65, Chapter XIV, §19, Theorem 2])

$$\alpha = \omega^{\beta_1}c_1 + \omega^{\beta_2}c_2 + \cdots + \omega^{\beta_k}c_k,$$

where $\beta_1 > \beta_2 > \cdots > \beta_k \geq 0$ are ordinals and c_i are positive integers, then the Cantor normal form of $\alpha \cdot \omega$ is ω^{β_1+1} (cf. [Sie65, Chapter XIV, §19, Exercise 4]). By Kedlaya's prediction, we split the proof into 3 cases:

1. If $\alpha \cdot \omega$ is finite, then clearly $\alpha = 0$;
2. If $\alpha \cdot \omega = \omega$, then $\beta_1 = 0$, and consequently $\alpha = c_1$ is finite;
3. If $\alpha \cdot \omega = \omega^\omega$, then $\omega^{\beta_1+1} = \omega^\omega$, which is impossible since ω is not a successor ordinal.

□

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